LECTURE NOTES ON

Applied Elasticity and Plasticity

PREPARED BY

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Syllabus

Applied Elasticity and Plasticity (3-1-0)

Module-I (14 hours)

Stress-strain relations for linearly elastic solids, Generalized Hooke's law. Analysis of three dimensional stresses and strains. Tensor character of stress. Strain-displacement relations, equilibrium equations, compatibility conditions and Airy's stress function,. Plane stress and plane strain, simple problems in cartesian and polar co-ordinates,

Module-II (13 hours)

Solution of axisymmetric problems, Bending of beams and plates, Kirkhhof and Mindlin concept. Torsion problem with St.Venant's approach-Prandtl's approach - Torsion of thin walled open and closed sections & thermal stress.

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Module-III (13 hours)

Theoretical concepts of plasticity, Yield criteria - Tresca and Von Mises criterion of yielding, Plastic stress strain relationship, Elastic plastic problems in bending and torsion.

Text Books

1. Timoshenko, S. and Goodier J.N. Theory of Elasticity, McGraw Hill Book Co., Newyork, 1988.

2. J. Chakrabarty, Theory of Plasticity, McGraw-Hill Book Company, New York 1990

Reference Books

1. Irving H.Shames and James,M.Pitarresi, Introduction to Solid Mechanics,Prentice Hall of India Pvt. Ltd., New Delhi -2002.

2.E.P. Popov, Engineering Mechanics of Solids, 2nd Ed., Prentice Hall India, 1998.

3 W.F.Chen and D.J.Han., Plasticity for structural Engineers., Springer-Verlag., NY., 1988.

4. Hoffman and Sachs, *Theory of Plasticity* - McGraw Hill., 2nd ed. 1985

5. Johnson and Mellor, *Engineering Plasticity*- Van-Nostrand., 1st edition, 1983

Module-I

Elasticity: All structural materials possess to a certain extent the property of elasticity i.e. if external forces, producing deformation of a structure, don't exceed a certain limit; the deformation disappears with the removal of the forces. In this course it will be assumed that the bodies undergoing the action of external forces are perfectly elastic, i.e. that they resume their initial form completely after removal of forces.

The simplest mechanical test consists of placing a standardized specimen with its ends in the grips of a tensile testing machine and then applying load under controlled conditions. Uniaxial loading conditions are thus approximately obtained. A force balance on a small element of the specimen yields the longitudinal (true) stress as

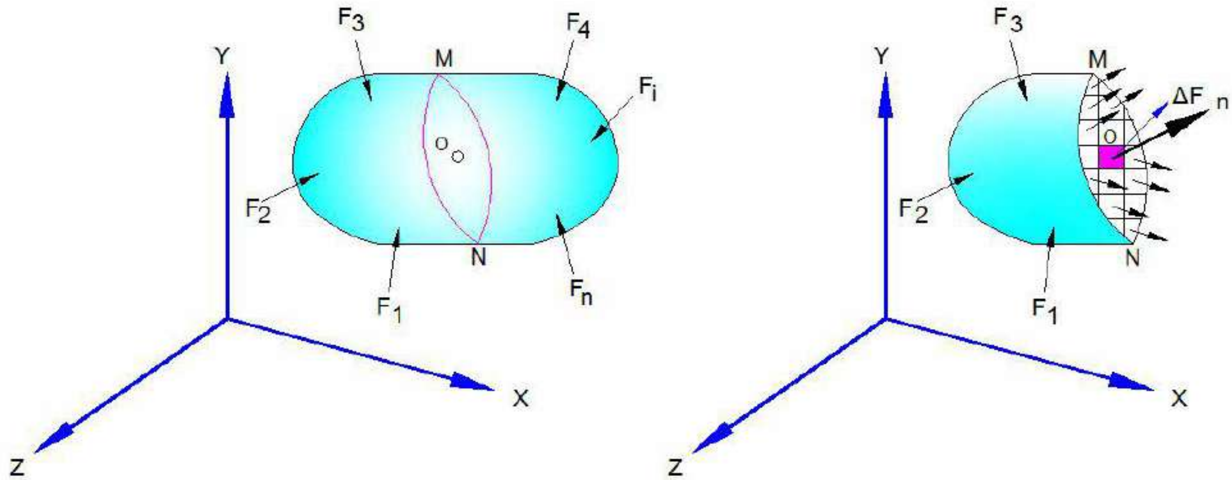
$$\sigma = \frac{F}{A}$$

Where, *F* is the applied force and *A* is the (instantaneous) cross sectional area of the specimen. Alternatively, if the initial cross sectional area A_0 is used, one obtains the engineering stress

$$\sigma_e = \frac{F}{A_0}$$

For loading in the elastic regime, for most engineering materials $\sigma_e = \sigma$

Stress: A body under the action of external forces, undergoes distortion and the effect due to this system of forces is transmitted throughout the body developing internal forces in it. To examine these internal forces at a point O in Figure (a), inside the body, consider a plane MN passing through the point O. If the plane is divided into a number of small areas, as in the Figure (b), and the forces acting on each of these are measured, it will be observed that these forces vary from one small area to the next. On the small area D*A* at point O, a force D*F* will be acting as shown in the Figure 2.1 (b). From this the concept of stress as the internal force per unit area can be understood. Assuming that the material is continuous, the term "stress" at any point across a small area ΔA can be defined by the limiting equation as below.



Force acting on a Body

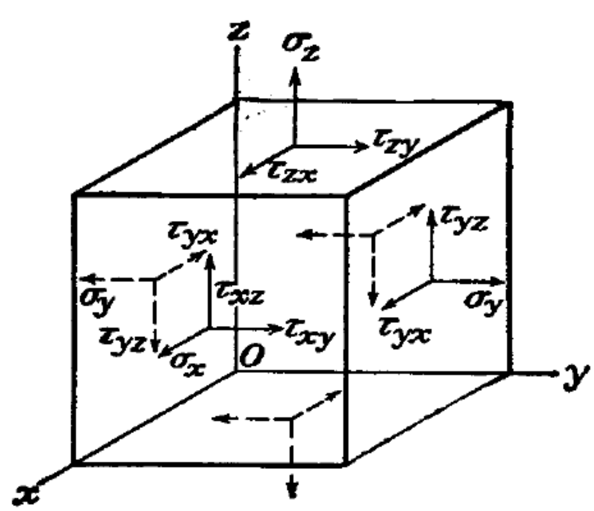
$$\text{Stress} = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

where ΔF is the internal force on the area ΔA surrounding the given point. Stress is sometimes referred to as force intensity.

Notation of Force and Stress: There are two kinds of external forces which may act on bodies force distribution over the surface of the bodies

face of the body, such as the pressure of one body on another, or hydrostatic pressure, are called *surface forces*. Forces distributed over the volume of a body, such as gravitational forces, magnetic forces, or in the case of a body in motion, inertia forces, are called *body forces*. The surface force per unit area we shall usually resolve into three components parallel to the coordinate axes and use for these components the notation $\bar{X}$, $\bar{Y}$, $\bar{Z}$. We shall also resolve the body force per unit volume into three components and denote these components by X, Y, Z.

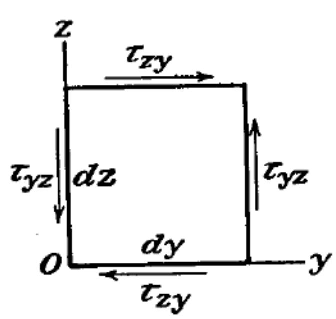
We shall use the letter σ for denoting normal stress and the letter τ for shearing stress. To indicate the direction of the plane on which the stress is acting, subscripts to these letters are used. If we take a very small cubic element at a point O, Fig. 1, with sides parallel to the coordinate axes, the notations for



the components of stress acting on the sides of this element and the directions taken as positive are as indicated. For the sides of element perpendicular to the y-axis, for instance, the normal components of stress acting on these sides are denoted by σ_y. The subscript y indicates that the stress is acting on a plane normal to the y-axis. The normal stress is taken positive when it produces tension and negative when it produces compression.

The shearing stress is resolved into two components parallel to the coordinate axes. Two subscript letters are used in this case, the first indicating the direction of the normal to the plane under consideration and the second indicating the direction of the component of the stress. For instance, if we again consider the sides perpendicular to the y-axis, the component in the x-direction is denoted by τ_{yx} and that in the z-direction by τ_{yz}. The positive directions of the components of shearing stress on any side of the cubic element are taken as the positive directions of the coordinate axes if a tensile stress on the same side would have the positive direction of the corresponding axis. If the tensile stress has a direction opposite to the positive axis, the positive direction of the shearing-stress components should be reversed. Following this rule the positive directions of all the components of stress acting on the right side of the cubic element (Fig. 3) coincide with the positive directions of the coordinate axes. The positive directions are all reversed if we are considering the left side of this element.

Components of Stress: **To describe the stress acting on the six sides of a cubic** element three symbols, $\sigma_x, \sigma_y, \sigma_z$, are necessary for normal stresses; and six symbols, $\tau_{xy}, \tau_{yx}, \tau_{xz}, \tau_{zx}, \tau_{yz}, \tau_{zy}$, for shearing stresses. By a simple consideration of the equilibrium of the element the number of symbols for shearing stresses can be reduced to three.



If we take the moments of the forces acting on the element about the x-axis, for instance, only the surface stresses shown in Fig. 4 need be considered. Body forces, such as the weight of the element, can be neglected in this instance, which **follows from the fact that in reducing one** dimensions of the element the body forces acting on it diminish as the cube of the linear dimensions while the surface forces diminish as the square of the linear dimensions. Hence, for a very small element, body forces are small quantities of higher order than surface forces and can be neglected in calculating the surface forces. Similarly, moments due to nonuniformity of distribution of normal forces are of higher order than those due to the shearing forces and vanish in the limit. Also the forces on each side can be considered to be the area of the side times the stress at the middle. Then denoting the dimensions of the small element in Fig. 4 by dx, dy, dz, the equation of equilibrium of this element, taking moments of forces about the x-axis, is

$$\tau_{zy}\, dx\, dy\, dz = \tau_{yz}\, dx\, dy\, dz$$

The two other equations can be obtained in the same manner. From these equations we find

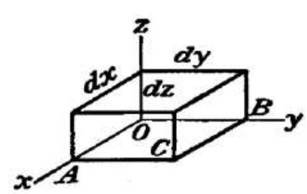
$$\tau_{xy} = \tau_{yx}, \qquad \tau_{zx} = \tau_{xz}, \qquad \tau_{zy} = \tau_{yz}$$

Hence for two perpendicular sides of a cubic elements the components of shearing stress perpendicular to the line of intersection of these sides are equal.

The six quantities $\sigma_x, \sigma_y, \sigma_z, \tau_{xy} = \tau_{yx}, \tau_{xz} = \tau_{zx}, \tau_{yz} = \tau_{zy}$ are therefore sufficient to describe the stresses acting on the coordinate planes through a point; these will be called the *components of stress* at the point.

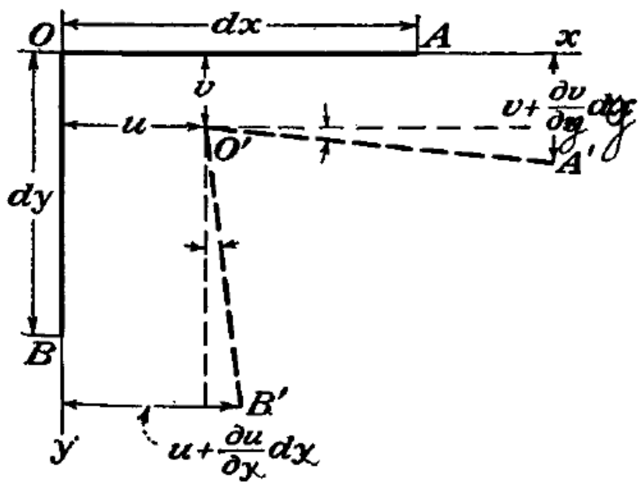
Components of strain:

the deformation of an elastic body it will be assumed that there are enough constraints to prevent the body from moving as a rigid body, so that no displacements of particles of the body are possible without a deformation of it.



In this book, only small deformations such as occur in engineering structures will be considered. The small displacements of particles of a deformed body will usually be resolved into components u, v, w parallel to the coordinate axes x, y, z, respectively. It will be assumed that these components are very small quantities varying continuously over the volume of the body. Consider a small element $dx\, dy\, dz$ of an **elastic body. If the body undergoes a deformation and** u, v, w

are the components of the displacement of the point O, the displacement in the x-direction of an adjacent point A on the x-axis is



$$u + \frac{\partial u}{\partial x} dx$$

due to the increase $(\partial u/\partial x)\, dx$ of the function u with increase of the coordinate x. The increase in length of the element OA due to deformation is therefore $(\partial u/\partial x)\, dx$. Hence the *unit elongation* at point O in the x-direction is $\partial u/\partial x$. In the same manner it can be shown that the unit elongations in the y- and z-directions are given by the derivatives $\partial v/\partial y$ and $\partial w/\partial z$.

Let us consider now the distortion of the angle between the elements OA and OB, Fig. 6. If u and v are the displacements of the point O in the x- and y-directions, the displacement of the point A in the y-direction and of the point B in the x-direction are $v + (\partial v/\partial x)\, dx$ and $u + (\partial u/\partial y)\, dy$, respectively. Due to these displacements the new direction $O'A'$ of the element OA is inclined to the initial direction by the small angle indicated in the figure, equal to $\partial v/\partial x$. In the same manner the direction $O'B'$ is inclined to OB by the small angle $\partial u/\partial y$. From this it will be seen that the initially right angle AOB between the two elements OA and OB is diminished by the angle $\partial v/\partial x + \partial u/\partial y$. This is the *shearing strain* between the planes xz and yz. The shearing strains between the planes xy and xz and the planes yx and yz can be obtained in the same manner.

We shall use the letter ϵ for unit elongation and the letter γ for unit shearing strain. To indicate the directions of strain we shall use the same subscripts to these letters as for the stress components. Then from the above discussion

$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \epsilon_y = \frac{\partial v}{\partial y}, \qquad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{2}$$

It will be shown later that, having the three unit elongations in three perpendicular directions and three unit shear strains related to the same directions, the elongation in *any* direction and the distortion of the angle between any two can be calculated later on.

The six quantities $\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{xz}\ and\ \gamma_{yz}$ are called the components of strain.

Generalized Hooke's Law:

Linear elastic behavior in the tension test is well described by Hooke's law, namely

$$\sigma = E\varepsilon$$

where *E* is the modulus of elasticity or Young's modulus. For most materials, this is a large number of the order of 10^{11} Pa.

The statement that the component of stress at a given point inside a linear elastic medium are linear homogeneous functions of the strain components at the point is known as the generalized Hooke's law. Mathematically, this implies that

$$\sigma^{ij} = D^{ijkl}\epsilon_{kl}$$

where σ^{ij} and ϵ_{ij} are, respectively the stress and strain tensor components. The quantity D^{ijkl} is the tensor of elastic constants and it characterizes the elastic properties of the medium. Since the stress tensor is symmetric, the elastic constants tensor consists of 36 components.

The elastic strain energy W is defined as the symmetric quadratic form

$$W = \frac{1}{2}\sigma^{ij}\epsilon_{kl} = \frac{1}{2}D^{ijkl}\epsilon_{ij}\epsilon_{kl}$$

and has the property that $\sigma^{ij} = \partial W/\partial\epsilon_{ij}$. Because of the symmetry of W, the actual number of elastic constants in the most general case is 21. This number is further reduced in special cases that are of much interest in applications. For instance, for isotropic materials (elastic properties the same in all directions) the number of elastic constants is 2. For orthotropic materials (characterized by three mutually perpendicular planes of symmetry) the number of constants is 9. If the material exhibits symmetry with respect to only one plane, the number of constants is 13.

Stress-Strain relations for Isotropic-elastic solid:

The generalized Hooke's law for isotropic solids is

$$\sigma_{\alpha\alpha} = 3K\epsilon_{\alpha\alpha}$$
$$\sigma'_{ij} = 2G\epsilon'_{ij}$$

where K and G are the elastic constants bulk modulus and shear modulus, respectively and the primes denote the stress and strain deviators.

Combination of the above with the definition of stress and strain deviation tensors yields the following commonly used forms of Hooke's law; for stress, in terms of strain

$$\sigma_{ij} = \lambda\epsilon_{\alpha\alpha}\delta_{ij} + 2G\epsilon_{ij}$$

and for strain, in terms of stress

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{\alpha\alpha}\delta_{ij}$$

The constants λ and G are called Lame's constants, while E is Young's modulus and ν is Poisson's ratio. Any of the above elastic constants can be expressed in terms of the others and only two are independent. Values of the above elastic constants for a wide variety of engineering materials are readily available in handbooks.

For an isotropic elastic solid in a rectangular Cartesian system of coordinates, the constitutive equations of behavior then become

$$\epsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]$$
$$\epsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})]$$
$$\epsilon_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]$$

$$\epsilon_{xy} = \frac{1}{2G}\sigma_{xy}$$
$$\epsilon_{yz} = \frac{1}{2G}\sigma_{yz}$$
$$\epsilon_{zx} = \frac{1}{2G}\sigma_{zx}$$

this formula reduced to its simplest form as

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$
$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

Stress-Strain relations for Anisotropic-elastic solid:

It is conventional in studying elastic deformation of anisotropic materials to relabel the six stress and strain components as follows:

$$\sigma_{11} = \sigma_1$$

$$\sigma_{22} = \sigma_2$$
$$\sigma_{33} = \sigma_3$$
$$\sigma_{23} = \sigma_4$$
$$\sigma_{13} = \sigma_5$$
$$\sigma_{12} = \sigma_6$$

$$\epsilon_{11} = \epsilon_1$$
$$\epsilon_{22} = \epsilon_2$$
$$\epsilon_{33} = \epsilon_3$$
$$\epsilon_{23} = \frac{1}{2}\epsilon_4$$
$$\epsilon_{13} = \frac{1}{2}\epsilon_5$$
$$\epsilon_{12} = \frac{1}{2}\epsilon_6$$

With the new notation and using the summation convention, Hooke's law becomes

$$\sigma_i = C_{ij}\epsilon_j$$

or equivalently

$$\epsilon_i = S_{ij}\sigma_j$$

where C_{ij} and S_{ij} are, respectively the elastic stiffness and compliance matrices. Depending on the symmetries existing in the material, only a few components of the above matrices are nonzero. For instance, for single crystals with cubic structure only C_{11}, C_{12} and C_{44} are nonzero. Values of the components of the above matrices for a variety of anisotropic materials are readily available in handbooks.

shearing strain and shearing stress.

Let us consider the particular case of deformation of the rectangular parallelopiped in which $\sigma_y = -\sigma_z$ and $\sigma_x = 0$. Cutting out an element $abcd$ by planes parallel to the x-axis and at 45 deg. to the y- and z-axes (Fig. 7), it may be seen from Fig. 7b, by summing up the forces along and perpendicular to bc, that the normal stress on the sides of this element is zero and the shearing stress on the sides is

$$\tau = \tfrac{1}{2}(\sigma_z - \sigma_y) = \sigma_z \qquad (c)$$

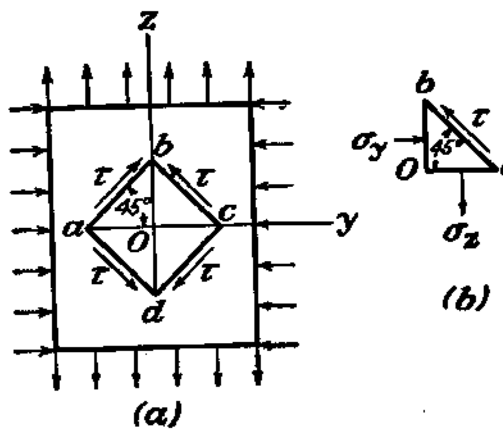


FIG. 7.

Such a condition of stress is called *pure shear*. The elongation of the vertical element Ob is equal to the shortening of the horizontal elements Oa and Oc, and neglecting a small quantity of the second order we conclude that the lengths ab and bc of the element do not change during deformation. The angle between the sides ab and bc changes, and the corresponding magnitude of shearing strain γ may be found from the triangle Obc. After deformation, we have

$$\frac{Oc}{Ob} = \tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = \frac{1 + \epsilon_y}{1 + \epsilon_z}$$

Substituting, from Eqs. (3),

$$\epsilon_z = \frac{1}{E}(\sigma_z - \nu\sigma_y) = \frac{(1 + \nu)\sigma_z}{E}$$

$$\epsilon_y = -\frac{(1 + \nu)\sigma_z}{E}$$

and noting that for small γ

$$\tan\left(\frac{\pi}{4}-\frac{\gamma}{2}\right)=\frac{\tan\frac{\pi}{4}-\tan\frac{\gamma}{2}}{1+\tan\frac{\pi}{4}\tan\frac{\gamma}{2}}=\frac{1-\frac{\gamma}{2}}{1+\frac{\gamma}{2}}$$

we find

$$\gamma=\frac{2(1+\nu)\sigma_z}{E}=\frac{2(1+\nu)\tau}{E} \tag{4}$$

Thus the relation between shearing strain and shearing stress is defined by the constants E and ν. Often the notation

$$G=\frac{E}{2(1+\nu)} \tag{5}$$

is used. Then Eq. (4) becomes

$$\gamma=\frac{\tau}{G}$$

The constant G, defined by (5), is called the *modulus of elasticity in shear* or the *modulus of rigidity*.

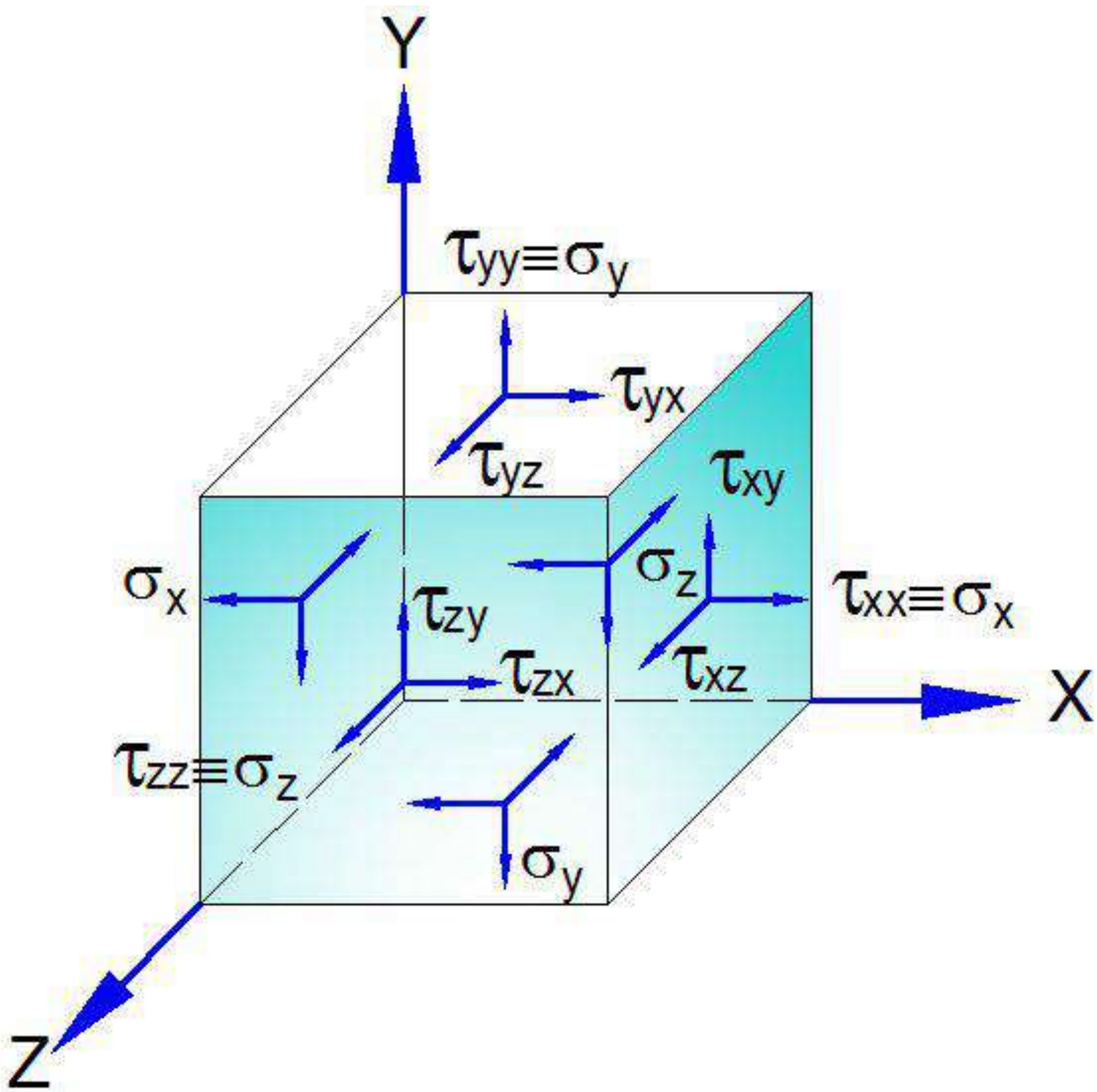
If shearing stresses act on the sides of an element, as shown in Fig. 3, the distortion of the angle between any two coordinate axes depends only on shearing-stress components parallel to these axes and we obtain

$$\gamma_{xy}=\frac{1}{G}\tau_{xy}, \qquad \gamma_{yz}=\frac{1}{G}\tau_{yz}, \qquad \gamma_{zx}=\frac{1}{G}\tau_{zx} \tag{6}$$

The elongations (3) and the distortions (6) are independent of each other. Hence the general case of strain, produced by three normal and three shearing components of stress, can be obtained by superposition: on the three elongations given by Eqs. (3) are superposed three shearing strains given by Eqs. (6).

Stress tensor:

Let O be the point in a body shown in Figure 2.1 (a). Passing through that point, infinitely many planes may be drawn. As the resultant forces acting on these planes is the same, the stresses on these planes are different because the areas and the inclinations of these planes are different. Therefore, for a complete description of stress, we have to specify not only its magnitude, direction and sense but also the surface on which it acts. For this reason, the stress is called a "Tensor".



Stress components acting on parallelopiped

Figure above depicts three-orthogonal co-ordinate planes representing a parallelopiped on which are nine components of stress. Of these three are direct stresses and six are shear stresses. In tensor notation, these can be expressed by the tensor t*ij*, where *i = x, y, z* and *j = x, y, z.*

In matrix notation, it is often written as

It is also written as

$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \qquad S = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

SPHERICAL AND DEVIATORIAL STRESS TENSORS

A general stress-tensor can be conveniently divided into two parts as shown above. Let us now define a new stress term (σ_m) as the mean stress, so that

$$\sigma_m = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

Imagine a hydrostatic type of stress having all the normal stresses equal to σ_m, and all the shear stresses are zero. We can divide the stress tensor into two parts, one having only the "hydrostatic stress" and the other, "deviatorial stress". The hydrostatic type of stress is given by

$$\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$

The deviatorial type of stress is given by

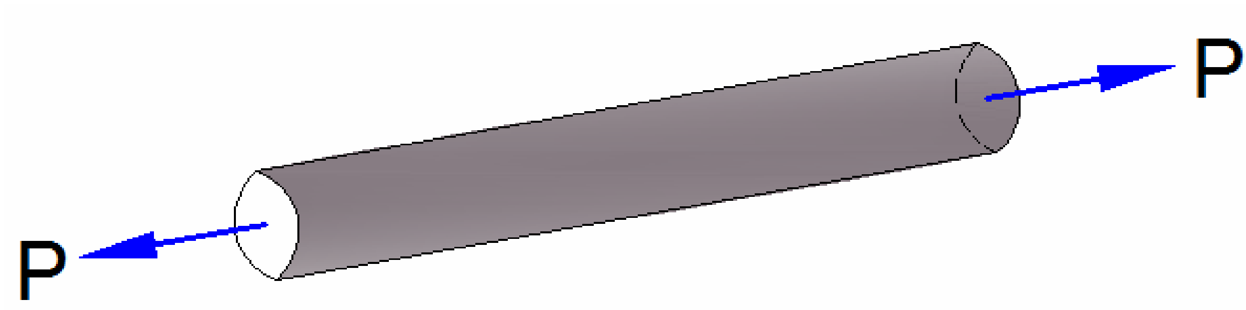
$$\begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_m \end{bmatrix}$$

Here the hydrostatic type of stress is known as "spherical stress tensor" and the other is known as the "deviatorial stress tensor".

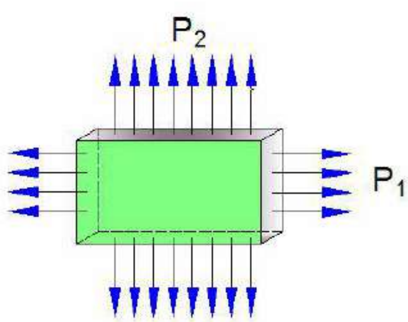
It will be seen later that the deviatorial part produces changes in shape of the body and finally causes failure. The spherical part is rather harmless, produces only uniform volume changes without any change of shape, and does not necessarily cause failure.

TYPES OF STRESS

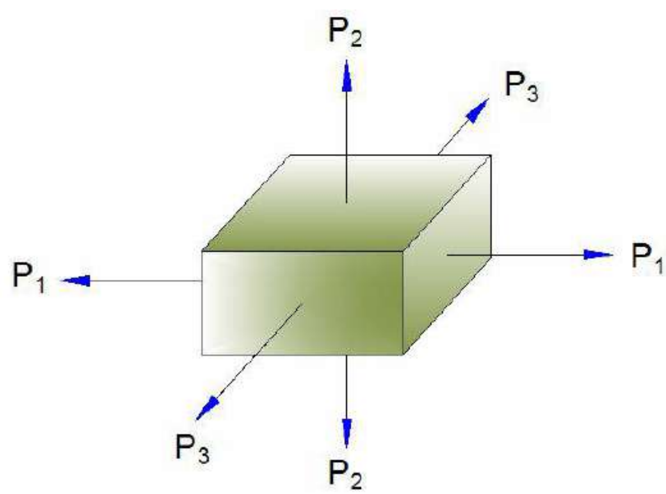
Stresses may be classified in two ways, i.e., according to the type of body on which they act, or the nature of the stress itself. Thus stresses could be one-dimensional, two-dimensional or three-dimensional as shown in the Figure (a), (b) and (c).



(a) One-dimensional Stress



(b) Two-dimensional Stress



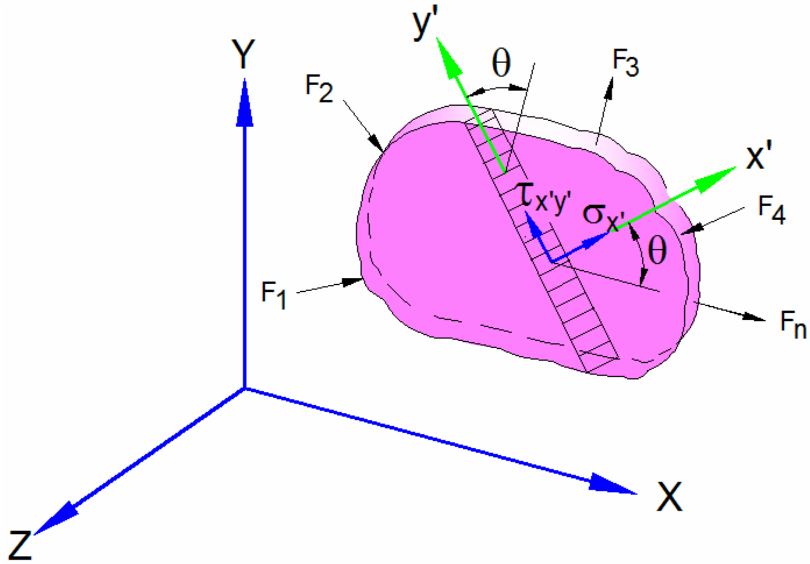
(c) Three-dimensional Stress

Types of Stress

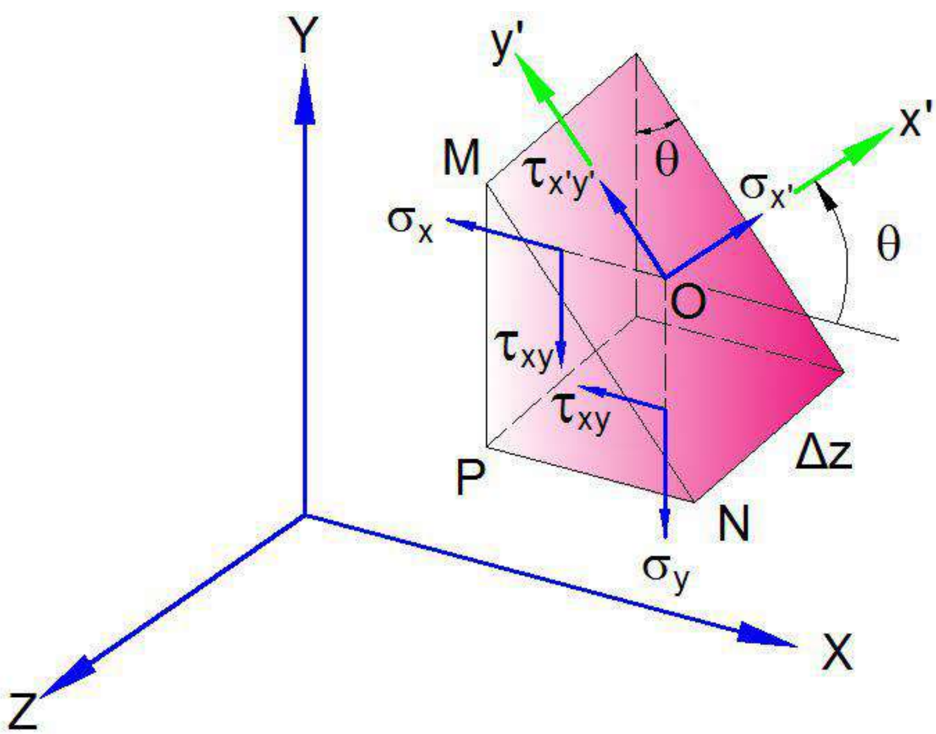
TWO-DIMENSIONAL STRESS AT A POINT

A two-dimensional state-of-stress exists when the stresses and body forces are independent of one of the co-ordinates. Such a state is described by stresses σ_x, σ_y and τ_{xy} and the X and Y body forces (Here *z* is taken as the independent co-ordinate axis).

We shall now determine the equations for transformation of the stress components σ_x, σ_y and τ_{xy} at any point of a body represented by infinitesimal element as shown in the Figure



Thin body subjected to stresses in xy plane



Stress components acting on faces of a small wedge cut from body of Figure shown above.

Consider an infinitesimal wedge as shown in Fig. cut from the loaded body in Figure above It is required to determine the stresses $\sigma_{x'}$, and $\tau_{x'y'}$, that refer to axes to axis x and y making an angle θ with axes *X*, Y as shown in the Figure. Let side MN is normal to the *x'* axis.

Considering $\sigma_{x'}$, and $\tau_{x'y'}$ as positive and area of side MN as unity, the sides MP and PN have areas cosθ and sinθ, respectively.

Equilibrium of the forces in the *x* and *y* directions requires that

$T_x = \sigma_{x'} \cos\theta + \tau_{x'y'} \sin\theta$

$T_y = \tau_{x'y'} \cos\theta + \sigma_{x'} \sin\theta$

where T_x and T_y are the components of stress resultant acting on MN in the x and y directions respectively. The normal and shear stresses on the x' plane (MN plane) are obtained by projecting T_x and T_y in the x' and y' directions.

$$\sigma_{x'} = T_x \cos\theta + T_y \sin\theta$$

$$\tau_{x'y'} = T_y \cos\theta - T_x \sin\theta$$

Substituting the stress resultant the above equations become

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

$$\tau_{x'y'} = \tau_{xy} \ (\cos^2\theta - \sin^2\theta) + (\sigma_y - \sigma_x) \sin\theta \cos\theta$$

The stress $\sigma_{y'}$ is obtained by substituting $\left(\theta + \frac{\pi}{2}\right)$ for θ in the expression for $\sigma_{x'}$.

By means of trigonometric identities

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta), \ \sin\theta \cos\theta = \frac{1}{2}\sin 2\theta,$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

The transformation equations for stresses are now written in the following form:

$$\sigma_{x'} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma_{y'} = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy} \cos 2\theta$$

Principal stress in two dimensions:

To ascertain the orientation of $x'y'$ corresponding to maximum or minimum $\sigma_{x'}$, the necessary condition $\frac{d\sigma_{x'}}{d\theta} = 0$, is applied to equation of yielding

$$-(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

Therefore, $\tan 2\theta = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$

As $2\theta = \tan(\pi + 2\theta)$, two directions, mutually perpendicular, are found to satisfy equation These are the principal directions along which the principal or maximum and minimum normal stress act.

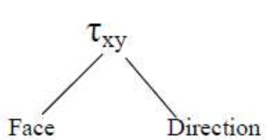
A principal plane is thus a plane on which the shear stress is zero. The principal stresses are determined by the equation

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Algebraically, larger stress given above is the maximum principal stress, denoted by σ_1. The minimum principal stress is represented by σ_2.

Analysis of three dimensional stresses and strains

Consider a cube of infinitesimal dimensions shown in figure; all stresses acting on this cube are identified on the diagram. The subscripts (τ) are the shear stress, associate the stress with a plane perpendicular to a given axis, the second designate the direction of the stress, i.e.



The stress symbols in figure (1), shows that three normal stresses: - $\sigma_x = \tau_{xx}$, $\sigma_y = \tau_{yy}$, $\sigma_z = \tau_{zz}$ and six shearing stresses, τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, τ_{zy}.

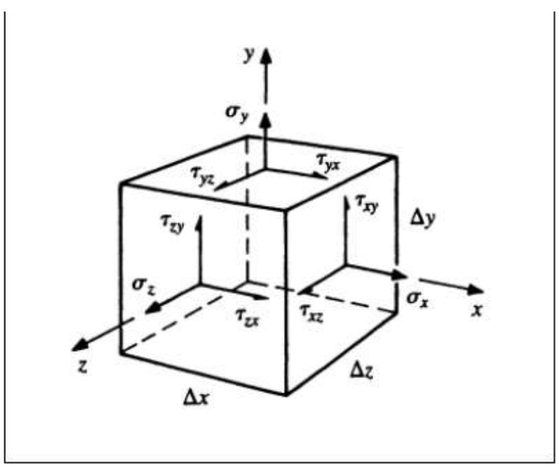


Figure (1)

The force vector (P) has only three components P_x , P_y and P_z.

$$\begin{matrix} P_x \\ P_y \\ P_z \end{matrix}$$

And stress vector: -

$$\begin{matrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{matrix} = \begin{matrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{matrix}$$

This is a matrix representation of the stress tensor. It is a second –rank tensor requiring two indices to identify its elements or components. A vector is first- rank tensor, and scalar is a zero tensor.

Sometimes, for brevity, a stress tensor is written in identical natation as τ_{ij} , where i, j and k designations x, y and z.

The stress tensor is symmetric, i.e. $\tau_{ij} = \tau_{ji}$ or

$$\tau_{xy} = \tau_{yx}$$

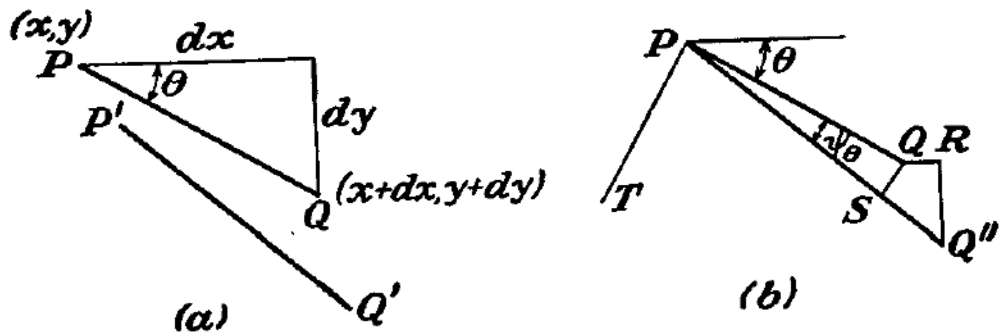
$$\tau_{xz} = \tau_{zx} \qquad \text{...2.2}$$

$$\tau_{yz} = \tau_{zy}$$

Strain-displacement relations:

When the strain components ϵ_x, ϵ_y and γ_{xy} at a point are known, the unit elongation or displacement for any direction and the decrease of a right angle (the shearing strain) of the any orientationat the point can be found. A line PQ as shown in figure below between the points (x,y) , (x+dx, y+dy) is translated, stretched (contracted) and rotated into the line element P'Q' when the deformation occurs. The displacement component of P are u,v those of Q are

$$u + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \qquad v + \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$



If $P'Q'$ in Fig. 17a is now translated so that P' is brought back to P, it is in the position PQ'' of Fig. 17b, and QR, RQ'' represent the components of the displacement of Q relative to P. Thus

$$QR = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \qquad RQ'' = \frac{\partial v}{\partial x} dx + \frac{\partial v}{dy} \partial y \tag{a}$$

The components of this relative displacement QS, SQ'', normal to PQ'' and along PQ'', can be found from these as

$$QS = -QR \sin\theta + RQ'' \cos\theta, \qquad SQ'' = QR \cos\theta + RQ'' \sin\theta \tag{b}$$

ignoring the small angle QPS in comparison with θ. Since the short line QS may be identified with an arc of a circle with center P, SQ'' gives the stretch of PQ. The unit elongation of $P'Q'$, denoted by ϵ_θ, is SQ''/PQ. Using (b) and (a) we have

$$\epsilon_\theta = \cos\theta\left(\frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}\right) + \sin\theta\left(\frac{\partial v}{\partial x}\frac{dx}{ds} + \frac{\partial v}{\partial y}\frac{dy}{ds}\right)$$

$$= \frac{\partial u}{\partial x}\cos^2\theta + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sin\theta\cos\theta + \frac{\partial v}{\partial y}\sin^2\theta$$

or

$$\epsilon_\theta = \epsilon_x \cos^2\theta + \gamma_{xy}\sin\theta\cos\theta + \epsilon_y \sin^2\theta \tag{c}$$

which gives the unit elongation for any direction θ.

The angle ψ_θ through which PQ is rotated is QS/PQ. Thus from (b) and (a),

$$\psi_\theta = -\sin\theta\left(\frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}\right) + \cos\theta\left(\frac{\partial v}{\partial x}\frac{dx}{ds} + \frac{\partial v}{\partial y}\frac{dy}{ds}\right)$$

or

$$\psi_\theta = \frac{\partial v}{\partial x}\cos^2\theta + \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\sin\theta\cos\theta - \frac{\partial u}{\partial y}\sin^2\theta \qquad (d)$$

The line element PT at right angles to PQ makes an angle $\theta + (\pi/2)$ with the x-direction, and its rotation $\psi_{\theta+\frac{\pi}{2}}$ is therefore given by (d) when $\theta + (\pi/2)$ is substituted for θ. Since $\cos[\theta + (\pi/2)] = -\sin\theta$, $\sin[\theta + (\pi/2)] = \cos\theta$, we find

$$\psi_{\theta+\frac{\pi}{2}} = \frac{\partial v}{\partial x}\sin^2\theta - \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\sin\theta\cos\theta - \frac{\partial u}{\partial y}\cos^2\theta \qquad (e)$$

The shear strain γ_θ for the directions PQ, PT is $\psi_\theta - \psi_{\theta+\frac{\pi}{2}}$, so

$$\gamma_\theta = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)(\cos^2\theta - \sin^2\theta) + \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)2\sin\theta\cos\theta$$

or

$$\tfrac{1}{2}\gamma_\theta = \tfrac{1}{2}\gamma_{xy}(\cos^2\theta - \sin^2\theta) + (\epsilon_y - \epsilon_x)\sin\theta\cos\theta \qquad (f)$$

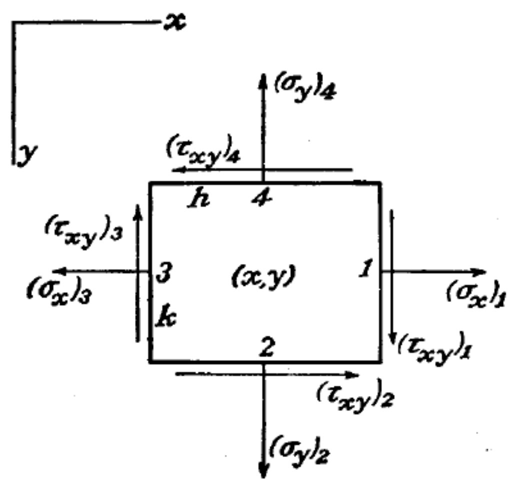
Comparing (c) and (f) with (13), we observe that they may be obtained from (13) by replacing σ by ϵ_θ, τ by $\gamma_\theta/2$, σ_x by ϵ_x, σ_y by ϵ_y, τ_{xy} by $\gamma_{xy}/2$, and α by θ. Consequently for each deduction made from (13) as to σ and τ, there is a corresponding deduction from (c) and (f) as to ϵ_θ and $\gamma_\theta/2$. Thus there are two values of θ, differing by 90 deg., for which γ_θ is zero. They are given by

$$\frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} = \tan 2\theta$$

The corresponding strains ϵ_θ are *principal strains*. A Mohr circle diagram analogous to Fig. 13 or Fig. 16 may be drawn, the ordinates representing $\gamma_\theta/2$ and the abscissas ϵ_θ. The principal strains ϵ_1, ϵ_2 will be the algebraically greatest and least values of ϵ_θ as a function of θ. The greatest value of $\gamma_\theta/2$ will be represented by the radius of the circle. Thus the greatest shearing strain $\gamma_{\theta\,\max.}$ is given by

$$\gamma_{\theta\,\max.} = \epsilon_1 - \epsilon_2$$

Equilibrium equations



Consider the equilibrium of a small rectangular block of edges h,k and unity as shown in the figure.

The stresses acting on the faces 1,2,3,4 and their positive directions are indicated in the figure. On account of the deviation of stress throughout the material, the value of, for instance, σ_x is not quite the same for face 1 as for face 3. The symbols σ_x, σ_y, τ_{xy} refer to the point x, y, the mid-point of the rectangle in Fig. 19. The values at the mid-points of the faces are denoted by $(\sigma_x)_1$, $(\sigma_x)_3$, etc. Since the faces are very small, the corresponding forces are obtained by multiplying these values by the areas of the faces on which they act.[1]

The body force on the block, which was neglected as a small quantity of higher order in considering the equilibrium of the triangular prism of Fig. 12, must be taken into consideration, because it is of the same order of magnitude as the terms due to the variations of the stress components which are now under consideration. If X, Y denote the components of body force per unit volume, the equation of equilibrium for forces in the x-direction is

$$(\sigma_x)_1 k - (\sigma_x)_3 k + (\tau_{xy})_2 h - (\tau_{xy})_4 h + Xhk = 0$$

or, dividing by hk,

$$\frac{(\sigma_x)_1 - (\sigma_x)_3}{h} + \frac{(\tau_{xy})_2 - (\tau_{xy})_4}{k} + X = 0$$

If now the block is taken smaller and smaller, *i.e.*, $h \to 0$, $k \to 0$, the limit of $[(\sigma_x)_1 - (\sigma_x)_3]/h$ is $\partial\sigma_x/\partial x$ by the definition of such a derivative.

Similarly $[(\tau_{xy})_2 - (\tau_{xy})_4]/k$ becomes $\partial\tau_{xy}/\partial y$. The equation of equilibrium for forces in the y-direction is obtained in the same manner. Thus

$$\begin{aligned} \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + X &= 0 \\ \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + Y &= 0 \end{aligned} \tag{18}$$

In practical applications the weight of the body is usually the only body force. Then, taking the y-axis downward and denoting by ρ the mass per unit volume of the body, Eqs. (18) become

$$\begin{aligned} \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} &= 0 \\ \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + \rho g &= 0 \end{aligned} \tag{19}$$

These are the differential equations of equilibrium for two-dimensional problems.

Boundary Conditions: Equations (18) or (19) must be satisfied at all points throughout the volume of the body. The stress components vary over the volume of the plate, and when we arrive at the boundary they must be such as to be in equilibrium with the external forces on the boundary of the plate, so that external forces may be regarded as a continuation of the internal stress distribution. These conditions of equilibrium at the boundary can be obtained from Eqs. (12). Taking the small triangular prism OBC (Fig. 12), so that the side BC coincides with the boundary of the plate, as shown in Fig. 20,

and denoting by $\bar{X}$ and $\bar{Y}$ the components of the surface forces per unit area at this point of the boundary, we have

$$\begin{aligned} \bar{X} &= l\sigma_x + m\tau_{xy} \\ \bar{Y} &= m\sigma_y + l\tau_{xy} \end{aligned} \tag{20}$$

in which l and m are the direction cosines of the normal N to the boundary.

In the particular case of a rectangular plate the coordinate axes are usually taken parallel to the sides of the plate and the boundary conditions (20) can be simplified. Taking, for instance, a side of the plate parallel to the x-axis we have for this part of the boundary the normal N parallel to the y-axis; hence $l = 0$ and $m = \pm 1$. Equations (20) then become

$$\bar{X} = \pm\tau_{xy}, \qquad \bar{Y} = \pm\sigma_y$$

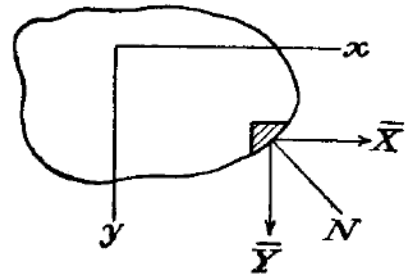


FIG. 20.

Here the positive sign should be taken if the normal N has the positive direction of the y-axis and the negative sign for the opposite direction of N. It is seen from this that at the boundary the stress components become equal to the components of the surface forces per unit area of the boundary.

Compatibility conditions: The problem of the theory of elasticity usually is to determine the state of stress in a body submitted to the action of given forces. In the case of a two-dimensional problem it is necessary to solve the differential equations of equilibrium (18), and the solution must be such as to satisfy the boundary conditions (20). These equations, derived by application of the equations of statics for absolutely rigid bodies, and containing three stress components σ_x, σ_y, τ_{xy}, are not sufficient for the determination of these components. The problem is a statically indeterminate one, and in order to obtain the solution the elastic deformation of the body must also be considered.

The mathematical formulation of the condition for compatibility of stress distribution with the existence of continuous functions u, v, w defining the deformation will be obtained from Eqs. (2). In the case of two-dimensional problems only three strain components need be considered, namely,

$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \epsilon_y = \frac{\partial v}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{a}$$

These three strain components are expressed by two functions u and v; hence they cannot be taken arbitrarily, and there exists a certain relation between the strain components which can easily be obtained from (a). Differentiating the first of the Eqs. (a) twice with respect to y, the second twice with respect to x, and the third once with respect to x and once with respect to y, we find

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \, \partial y} \tag{21}$$

This differential relation, called the *condition of compatibility*, must be satisfied by the strain components to secure the existence of functions u and v connected with the strain components by Eqs. (a). By using Hooke's law, [Eqs. (3)], the condition (21) can be transformed into a relation between the components of stress.

In the case of plane stress distribution (Art. 7), Eqs. (3) reduce to

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y), \qquad \epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \tag{22}$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} \tag{23}$$

Substituting in Eq. (21), we find

$$\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) = 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \, \partial y} \tag{b}$$

This equation can be written in a different form by using the equations of equilibrium. For the case when the weight of the body is the only body force, differentiating the first of Eqs. (19) with respect to x and the second with respect to y and adding them, we find

$$2\frac{\partial^2 \tau_{xy}}{\partial x \, \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2}$$

Substituting in Eq. (b), the compatibility equation in terms of stress components becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0 \tag{24}$$

Proceeding in the same manner with the general equations of equilibrium (18) we find

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) \tag{25}$$

In the case of plane strain (Art. 8), we have

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

and from Hooke's law (Eqs. 3), we find

$$\epsilon_x = \frac{1}{E}[(1 - \nu^2)\sigma_x - \nu(1 + \nu)\sigma_y]$$
$$\epsilon_y = \frac{1}{E}[(1 - \nu^2)\sigma_y - \nu(1 + \nu)\sigma_x] \tag{26}$$

$$\gamma_{xy} = \frac{2(1 + \nu)}{E}\tau_{xy} \tag{27}$$

Substituting in Eq. (21), and using, as before, the equations of equilibrium (19), we find that the compatibility equation (24) holds also for plane strain. For the general case of body forces we obtain from Eqs. (21) and (18) the compatibility equation in the following form:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -\frac{1}{1 - \nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) \tag{28}$$

The equations of equilibrium (18) or (19) together with the boundary conditions (20) and one of the above compatibility equations give us a system of equations which is usually sufficient for the complete determination of the stress distribution in a two-dimensional problem.[1] The particular cases in which certain additional considerations are necessary will be discussed later (page 117). It is interesting to note that in the case of constant body forces the equations determining stress distribution do not contain the elastic constants of the material. Hence the stress distribution is the same for all isotropic materials, provided the equations are sufficient for the complete determination of the stresses. The conclusion is of practical importance: we shall see later that in the case of transparent materials, such as glass or xylonite, it is possible to determine stresses by an optical method using polarized light (page 131). From the above discussion it is evident that experimental results obtained with a transparent material in most cases can be applied immediately to any other material, such as steel.

It should be noted also that in the case of constant body forces the compatibility equation (24) holds both for the case of plane stress and for the case of plane strain. Hence the stress distribution is the same in these two cases, provided the shape of the boundary and the external forces are the same.[2]

Airy's stress function: It has been shown that a solution of two dimensional problems reduces to the integration of the differential equations of equilibrium together with the compatibility equation and the boundary conditions. If we begin with the case when the weight of the body is the only body force, the equations to be satisfied are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$
$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho g = 0 \qquad (a)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0 \qquad (b)$$

To these equations the boundary conditions (20) should be added. Thus usual method of solving these equations is by introducing a new function, called stress function. This function was introduced in the solution of two dimensional problems by G. B. Airy so it is also called as **Airy stress function**.

As is easily checked, the equations (a) are satisfied by taking any function ϕ of x and y and putting the following expressions for the stress components.

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} - \rho g y, \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} - \rho g y, \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x\, \partial y}$$

In this manner we can get a variety of solutions of the equations of equilibrium (a). The true solution of the problem is that which satisfies also the compatibility equation (b). Substituting the above expressions for the stress components into equation (b), we find that the stress function ϕ must satisfy the equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2\, \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

Thus the solution of a two-dimensional problem, when the weight of the body is the only body force, reduces to finding a solution of above equation, which satisfies the boundary conditions of the problem.

Let us now consider a more general case of body forces and assume that these forces have a potential. Then the components X and Y in Equation 18 are given by equations

$$X = -\frac{\partial V}{\partial x}$$
$$Y = -\frac{\partial V}{\partial y} \qquad (c)$$

in which V is the potential function. Equations (18) become

$$\frac{\partial}{\partial x}(\sigma_x - V) + \frac{\partial \tau_{xy}}{\partial y} = 0$$
$$\frac{\partial}{\partial y}(\sigma_y - V) + \frac{\partial \tau_{xy}}{\partial x} = 0$$

These equations are of the same form as Eqs. (a) and can be satisfied by taking

$$\sigma_x - V = \frac{\partial^2 \phi}{\partial y^2}, \qquad \sigma_y - V = \frac{\partial^2 \phi}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x\, \partial y}$$

In which ϕ is the stress function. Substituting above expressions in the compatibility equation for plane stress distribution we find

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2\, \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = -(1 - \nu)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

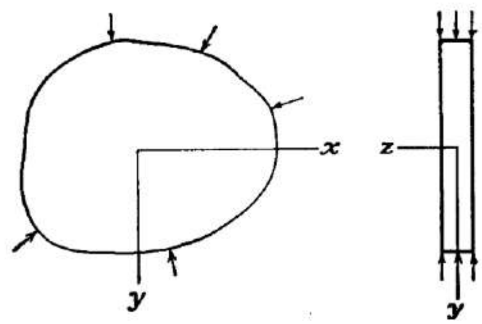
When the body force is simply the weight the potential V is –ρgh. In this case the right hand side of above equation reduced to zero. By taking the solution ϕ=0 the stree distribution can found out as

$$\sigma_x = -\rho g y, \qquad \sigma_y = -\rho g y, \qquad \tau_{xy} = 0$$

As a possible state of stress due to gravity. This is a state of hydrostatic pressure ρgh in two dimensions, with zero stress at Y=0. It can exist in a plate or cylinder of any shape provided the corresponding boundary force are applied.

Module-II

Plane stress: If a thin plate is loaded by forces applied at the boundary**,** parallel to the plane of the plate and distributed uniformly over the thickness, the stress components σ_x , τ_{xz} , τ_{yz} are zero on both faces of the plate and it may be assumed tentatively, that they are zero also within the plate. The state of stress is then specified by σ_x , σ_y , τ_{xy} only and is called *Plane stress*.



It may be assumed that these three components are independent of z i.e. they don't vary through thickness. They are functions of x and y only.

Plane strain: A similar simplification is possible at the other extreme when the dimension of the body is in the z-direction is very large.
If a long cylindrical or prismatical body is loaded by forces which are perpendicular to the longitudinal elements and do not vary along the length, it may be assumed that all cross sections are in the same condition. It is simplest to suppose at first that the end sections are confined between fixed smooth rigid planes, so that displacement in the axial direction is prevented. The effect of removing these will be examined later. Since there is no axial displacement at the ends, and, by symmetry, at the mid-section, it may be assumed that the same holds at every cross section.

There are many important problems of this kind—a retaining wall with lateral pressure (Fig. 9), a culvert or tunnel (Fig. 10), a cylindrical tube with internal pressure, a cylindrical roller compressed by forces in a diametral plane as in a roller bearing (Fig. 11). In each case of course the loading must not vary along the length. Since conditions are the same at all cross sections, it is sufficient to consider only a slice between two sections unit distance apart. The components u and v of the displacement are functions of x and y but are independent of the

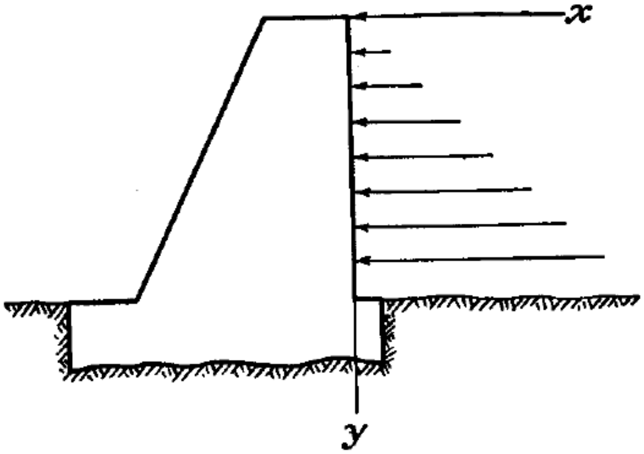


FIG. 9.

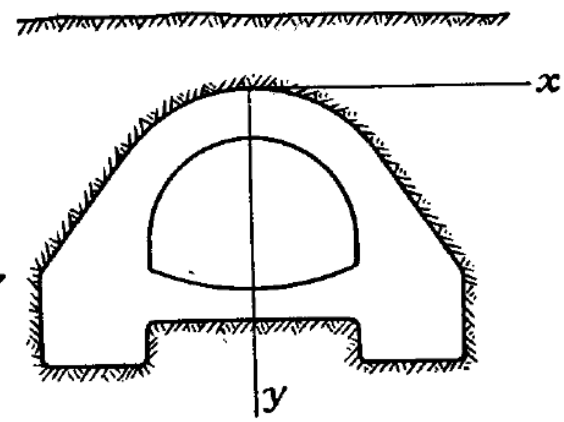


FIG. 10.

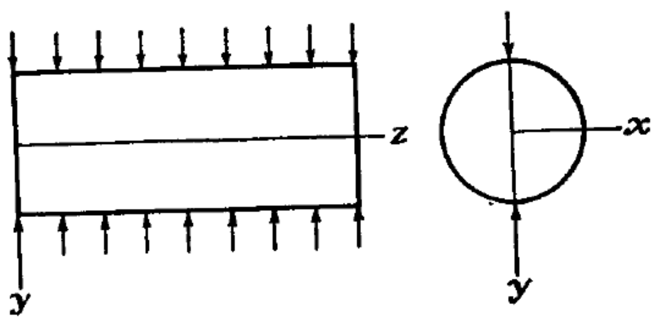


FIG. 11.

longitudinal coordinate z. Since the longitudinal displacement w is zero, Eqs. (2) give

$$
\begin{aligned}
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \qquad (a) \\
\epsilon_z &= \frac{\partial w}{\partial z} = 0
\end{aligned}
$$

The longitudinal normal stress σ_z can be found in terms of σ_x and σ_y by means of Hooke's law, Eqs. (3). Since $\epsilon_z = 0$ we find

$$\sigma_z - \nu(\sigma_x + \sigma_y) = 0$$

or

$$\sigma_z = \nu(\sigma_x + \sigma_y) \qquad (b)$$

These normal stresses act over the cross sections, including the ends, where they represent forces required to maintain the plane strain, and provided by the fixed smooth rigid planes.

By Eqs. (a) and (6), the stress components τ_{xz} and τ_{yz} are zero, and, by Eq. (b), σ_z can be found from σ_x and σ_y. Thus the plane strain problem, like the plane stress problem, reduces to the determination of σ_x, σ_y, and τ_{xy} as functions of x and y only.

Simple problems in cartesian and polar co-ordinates

Solution by polynomials: it has been shown that the solution of two-dimensional problems, when body forces are absent or are constant, is reduced to the integration of the differential equation

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\,\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 \qquad (a)$$

having regard to boundary conditions (20). In the case of long rectangular strips, solutions of Eq. (a) in the form of polynomials are of interest. By taking polynomials of various degrees, and suitably adjusting their coefficients, a number of practically important problems can be solved.[1]

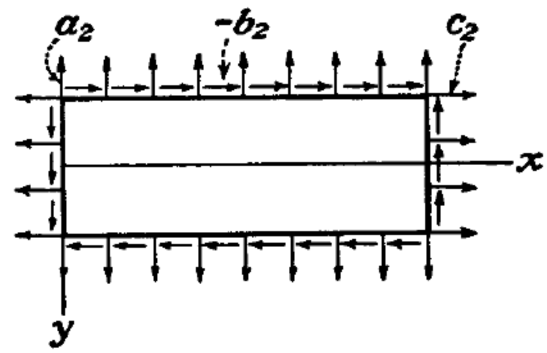


Fig. 21.

Beginning with a polynomial of the second degree

$$\phi_2 = \frac{a_2}{2}x^2 + b_2xy + \frac{c_2}{2}y^2 \qquad (b)$$

which evidently satisfies Eq. (a), we find from Eqs. (29), putting $\rho g = 0$,

$$\sigma_x = \frac{\partial^2\phi_2}{\partial y^2} = c_2, \qquad \sigma_y = \frac{\partial^2\phi_2}{\partial x^2} = a_2, \qquad \tau_{xy} = -\frac{\partial^2\phi_2}{\partial x\,\partial y} = -b_2$$

All three stress components are constant throughout the body, *i.e.*, the stress function (b) represents a combination of uniform tensions or

Let us consider now a stress function in the form of a polynomial of the third degree:

$$\phi_3 = \frac{a_3}{3 \cdot 2} x^3 + \frac{b_3}{2} x^2 y + \frac{c_3}{2} xy^2 + \frac{d_3}{3 \cdot 2} y^3 \tag{c}$$

This also satisfies Eq. (a). Using Eqs. (29) and putting $\rho g = 0$, we find

$$\sigma_x = \frac{\partial^2 \phi_3}{\partial y^2} = c_3 x + d_3 y$$

$$\sigma_y = \frac{\partial^2 \phi_3}{\partial x^2} = a_3 x + b_3 y$$

$$\tau_{xy} = -\frac{\partial^2 \phi_3}{\partial x \, \partial y} = -b_3 x - c_3 y$$

For a rectangular plate, taken as in Fig. 22, assuming all coefficients except d_3 equal to zero, we obtain pure bending. If only coefficient a_3 is different from zero, we obtain pure bending by normal stresses applied to the sides $y = \pm c$ of the plate. If coefficient b_3 or c_3 is taken

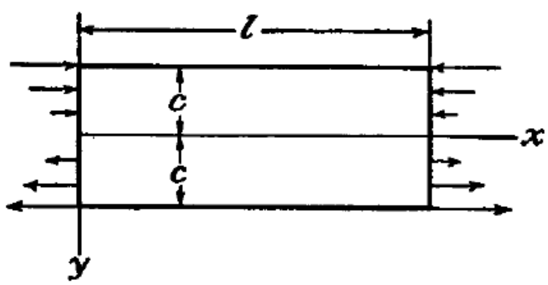


Fig. 22.

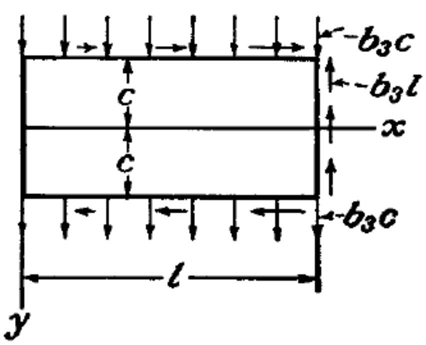


Fig. 23.

different from zero, we obtain not only normal but also shearing stresses acting on the sides of the plate. Figure 23 represents, for instance, the case in which all coefficients, except b_3 in function (c), are equal to zero. The directions of stresses indicated are for b_3 positive. Along the sides $y = \pm c$ we have uniformly distributed tensile and compressive stresses, respectively, and shearing stresses proportional to x. On the side $x = l$ we have only the constant shearing stress $-b_3 l$, and there are no stresses acting on the side $x = 0$. An analogous stress distribution is obtained if coefficient c_3 is taken different from zero.

distribution is obtained if coefficient c_3 is taken different from zero.

In taking the stress function in the form of polynomials of the second and third degrees we are completely free in choosing the magnitudes of the coefficients, since Eq. (a) is satisfied whatever values they may have. In the case of polynomials of higher degrees Eq. (a) is satisfied only if certain relations between the coefficients are satisfied. Taking, for instance, the stress function in the form of a polynomial of the fourth degree,

$$\phi_4 = \frac{a_4}{4 \cdot 3} x^4 + \frac{b_4}{3 \cdot 2} x^3 y + \frac{c_4}{2} x^2 y^2 + \frac{d_4}{3 \cdot 2} x y^3 + \frac{e_4}{4 \cdot 3} y^4 \qquad (d)$$

and substituting it into Eq. (a), we find that the equation is satisfied only if

$$e_4 = -(2c_4 + a_4)$$

The stress components in this case are

$$\sigma_x = \frac{\partial^2 \phi_4}{\partial y^2} = c_4 x^2 + d_4 xy - (2c_4 + a_4) y^2$$
$$\sigma_y = \frac{\partial^2 \phi_4}{\partial x^2} = a_4 x^2 + b_4 xy + c_4 y^2$$
$$\tau_{xy} = \frac{\partial^2 \phi_4}{\partial x \, \partial y} = -\frac{b_4}{2} x^2 - 2c_4 xy - \frac{d_4}{2} y^2$$

Coefficients $a_4, \ldots, d_4$ in these expressions are arbitrary, and by suitably adjusting them we obtain various conditions of loading of a rectangular plate. For instance, taking all coefficients except d_4 equal to zero, we find

$$\sigma_x = d_4 xy, \qquad \sigma_y = 0, \qquad \tau_{xy} = -\frac{d_4}{2} y^2 \qquad (e)$$

Assuming d_4 positive, the forces acting on the rectangular plate shown in Fig. 24 and producing the stresses (e) are as given. On the longitudinal sides $y = \pm c$ are uniformly distributed shearing forces; on the ends shearing forces are distributed according to a parabolic law. The shearing forces acting on the boundary of the plate reduce to the couple[1]

$$M = \frac{d_4c^2l}{2} \cdot 2c - \frac{1}{3}\frac{d_4c^2}{2} \cdot 2c \cdot l = \frac{2}{3} d_4c^3l$$

This couple balances the couple produced by the normal forces along the side $x = l$ of the plate.

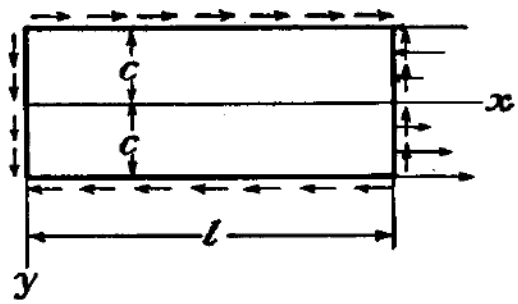


FIG. 24.

Let us consider a stress function in the form of a polynomial of the fifth degree.

Substituting in Eq. (a) we find that this equation is satisfied if

$$e_5 = -(2c_5 + 3a_5)$$
$$f_5 = -\tfrac{1}{3}(b_5 + 2d_5)$$

The corresponding stress components are:

$$\sigma_x = \frac{\partial^2\phi_5}{\partial y^2} = \frac{c_5}{3}x^3 + d_5x^2y - (2c_5 + 3a_5)xy^2 - \frac{1}{3}(b_5 + 2d_5)y^3$$

$$\sigma_y = \frac{\partial^2\phi_5}{\partial x^2} = a_5x^3 + b_5x^2y + c_5xy^2 + \frac{d_5}{3}y^3$$

$$\tau_{xy} = -\frac{\partial^2\phi_5}{\partial x\,\partial y} = -\frac{1}{3}b_5x^3 - c_5x^2y - d_5xy^2 + \frac{1}{3}(2c_5 + 3a_5)y^3$$

Again coefficients $a_5, \ldots, d_5$ are arbitrary, and in adjusting them we obtain solutions for various loading conditions of a plate. Taking,

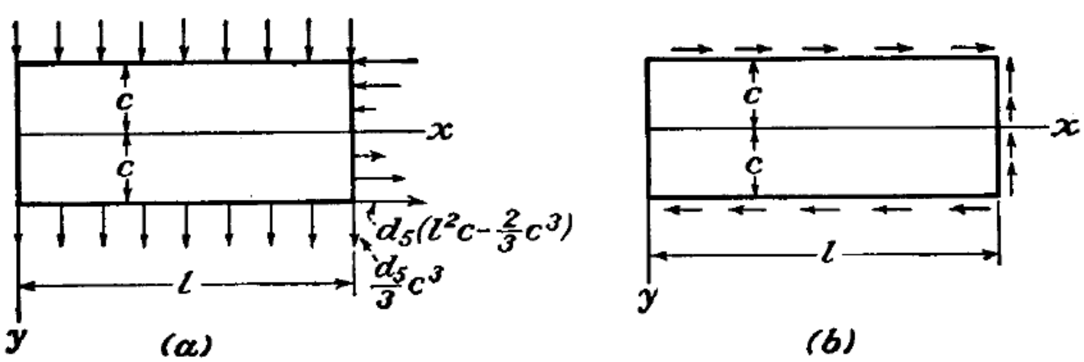


FIG. 25.

for instance, all coefficients, except d_5, equal to zero we find

$$\sigma_x = d_5(x^2y - \tfrac{2}{3}y^3)$$

$$\sigma_y = \tfrac{1}{3}d_5y^3 \qquad (g)$$
$$\tau_{xy} = -d_5xy^2$$

The normal forces are uniformly distributed along the longitudinal sides of the plate (Fig. 25*a*). Along the side $x = l$, the normal forces consist of two parts, one following a linear law and the other following the law of a cubic parabola. The shearing forces are proportional to x on the longitudinal sides of the plate and follow a parabolic law along the side $x = l$. The distribution of these stresses is shown in Fig. 25*b*.

Since Eq. (*a*) is a linear differential equation, it may be concluded that a sum of several solutions of this equation is also a solution. We can superpose the elementary solutions considered in this article and in this manner arrive at new solutions of practical interest. Several examples of the application of this method of superposition will be considered.

ST. VENANT'S PRINCIPLE

For the purpose of analysing the statics or dynamics of a body, one force system may be replaced by an equivalent force system whose force and moment resultants are identical. Such force resultants, while equivalent need not cause an identical distribution of strain, owing to difference in the arrangement of forces. St. Venant's principle permits the use of an equivalent loading for the calculation of stress and strain.

St. Venant's principle states that if a certain system of forces acting on a portion of the surface of a body is replaced by a different system of forces acting on the same portion of the body, then the effects of the two different systems at locations sufficiently far distant from the region of application of forces, are essentially the same, provided that the two systems of forces are statically equivalent (i.e., the same resultant force and the same resultant moment). St. Venant principle is very convenient and useful in obtaining solutions to many engineering problems in elasticity. The principle helps to the great extent in prescribing the boundary conditions very precisely when it is very difficult to do so.

Determination of Displacement: when the components of stress are found from the previous equations, the components of strain can be obtained by using Hooke's law, Eqs. (3) and (6). Then the displacements u and v can be obtained from the equations

$$\frac{\partial u}{\partial x} = \epsilon_x, \qquad \frac{\partial v}{\partial y} = \epsilon_y, \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy} \tag{a}$$

The integration of these equations in each particular case does not present any difficulty, and we shall have several examples of their application. It may be seen at once that the strain components (a) remain unchanged if we add to u and v the linear functions

$$u_1 = a + by, \qquad v_1 = c - bx \tag{b}$$

in which a, b, and c are constants. This means that the displacements are not entirely determined by the stresses and strains. On the displacements due to the internal strains a displacement like that of a rigid body can be superposed. The constants a and c in Eqs. (b) represent a translatory motion of the body and the constant b is a small angle of rotation of the rigid body about the z-axis.

It has been shown (see page 25) that in the case of constant body forces the stress distribution is the same for plane stress distribution or plane strain. The displacements however are different for these two problems, since in the case of plane stress distribution the components of strain, entering into Eqs. (a), are given by equations

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y), \qquad \epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x), \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

and in the case of plane strain the strain components are:

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] = \frac{1}{E}[(1 - \nu^2)\sigma_x - \nu(1 + \nu)\sigma_y]$$

$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] = \frac{1}{E}[(1 - \nu^2)\sigma_y - \nu(1 + \nu)\sigma_x]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

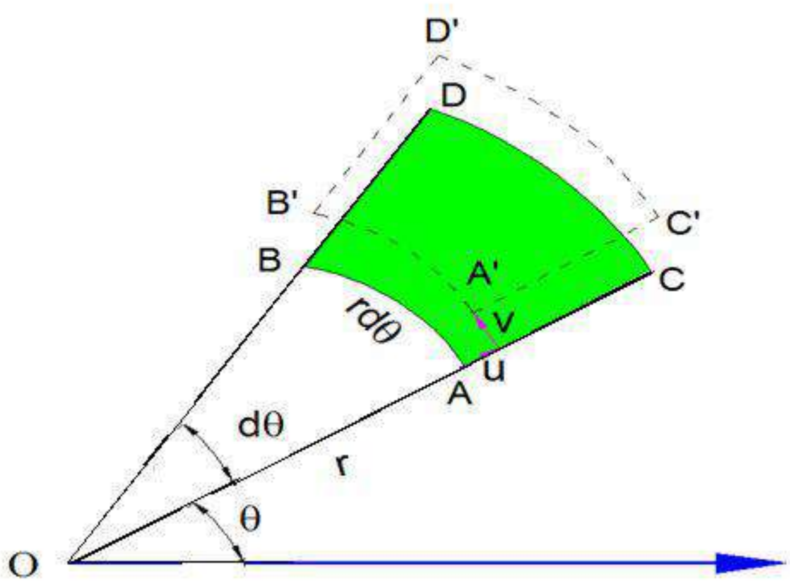
It is easily verified that these equations can be obtained from the preceding set for plane stress by replacing E in the latter by $E/(1 - \nu^2)$, and ν by $\nu/(1 - \nu)$. These substitutions leave G, which is $E/2(1 + \nu)$, unchanged.

Two Dimensional Problems in Polar Coordinate System

In any elasticity problem the proper choice of the co-ordinate system is extremely important since this choice establishes the complexity of the mathematical expressions employed to satisfy the field equations and the boundary conditions. In order to solve two dimensional elasticity problems by employing a polar co-ordinate reference frame, the equations of equilibrium, the definition of Airy's Stress function, and one of the stress equations of compatibility must be established in terms of Polar Co-ordinates.

STRAIN-DISPLACEMENT RELATIONS

Case 1: For Two Dimensional State of Stress



Deformed element in two dimensions

Consider the deformation of the infinitesimal element *ABCD*, denoting r and q displacements by *u* and *v* respectively. The general deformation experienced by an element may be regarded as composed of (1) a change in the length of the sides, and (2) rotation of the sides as shown in the figure above.

Referring to the figure, it is observed that a displacement *"u"* of side *AB* results in both radial and tangential strain.

Therefore, Radial strain = $\varepsilon_r = \dfrac{\partial u}{\partial r}$

and tangential strain due to displacement *u* per unit length of *AB* is

$$(\varepsilon_\theta)_u = \frac{(r+u)d\theta - rd\theta}{rd\theta} = \frac{u}{r}$$

Tangential strain due to displacement *v* is given by

$$(\varepsilon_\theta)_v = \frac{\left(\dfrac{\partial v}{\partial \theta}\right)d\theta}{rd\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Hence, the resultant strain is

$$\varepsilon_\theta = (\varepsilon_\theta)_u + (\varepsilon_\theta)_v$$

$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\left(\frac{\partial v}{\partial \theta}\right)$$

Similarly, the shearing strains can be calculated due to displacements *u* and *v* as below.

Component of shearing strain due to *u* is

$$(\gamma_{r\theta})_u = \frac{\left(\dfrac{\partial u}{\partial \theta}\right)d\theta}{rd\theta} = \frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right)$$

Component of shearing strain due to *v* is

$$(\gamma_{r\theta})_v = \frac{\partial v}{\partial r} - \left(\frac{v}{r}\right)$$

Therefore, the total shear strain is given by

$$\gamma_{r\theta} = (\gamma_{r\theta})_u + (\gamma_{r\theta})_v$$

$$\gamma_{r\theta} = \frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right) + \frac{\partial v}{\partial r} - \left(\frac{v}{r}\right)$$

Case 2: For three dimensional stress state

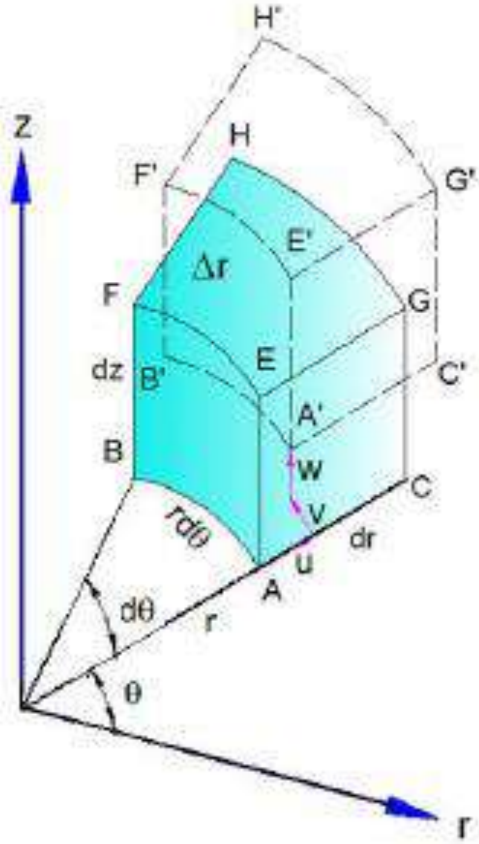


Figure 6.2 Deformed element in three dimensions

The strain-displacement relations for the most general state of stress are given by

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{1}{r}\left(\frac{\partial v}{\partial \theta}\right) + \left(\frac{u}{r}\right), \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right) - \left(\frac{v}{r}\right)$$

$$\gamma_{\theta z} = \frac{1}{r}\left(\frac{\partial w}{\partial \theta}\right) + \left(\frac{\partial v}{\partial z}\right)$$

$$\gamma_{zr} = \frac{\partial u}{\partial z} + \left(\frac{\partial w}{\partial r}\right)$$

Compatibility Equation:

We have from strain displacement relation

Radial strain, $\varepsilon_r = \dfrac{\partial u}{\partial r}$ (6.9a)

Tangential strain, $\varepsilon_\theta = \left(\dfrac{1}{r}\right)\dfrac{\partial v}{\partial \theta} + \left(\dfrac{u}{r}\right)$ (6.9b)

and total shearing strain, $\gamma_{r\theta} = \dfrac{\partial v}{\partial r} - \left(\dfrac{v}{r}\right) + \left(\dfrac{1}{r}\right)\dfrac{\partial u}{\partial \theta}$ (6.9c)

Differentiating Equation (6.9a) with respect to θ and Equation (6.9b) with respect to r, we get

$$\frac{\partial \varepsilon_r}{\partial \theta} = \frac{\partial^2 u}{\partial r \partial \theta} \quad (6.9d)$$

$$\frac{\partial \varepsilon_\theta}{\partial r} = \left(\frac{1}{r}\right)\frac{\partial u}{\partial r} - \left(\frac{1}{r^2}\right)u + \frac{1}{r}.\frac{\partial^2 v}{\partial r \partial \theta} - \left(\frac{1}{r^2}\right).\frac{\partial v}{\partial \theta}$$

$$= \frac{\varepsilon_r}{r} + \left(\frac{1}{r}\right)\frac{\partial^2 v}{\partial r \partial \theta} - \frac{1}{r}\left[\frac{u}{r} + \left(\frac{1}{r}\right)\frac{\partial v}{\partial \theta}\right]$$

$$\therefore \frac{\partial \varepsilon_\theta}{\partial r} = \frac{\varepsilon_r}{r} + \left(\frac{1}{r}\right)\frac{\partial^2 v}{\partial r \partial \theta} - \left(\frac{1}{r}\right)\varepsilon_\theta \quad (6.9e)$$

Now, Differentiating Equation (6.9c) with respect to r and using Equation (6.9d), we get

$$\frac{\partial \gamma_{r\theta}}{\partial r} = \frac{\partial^2 v}{\partial r^2} - \left(\frac{1}{r}\right)\frac{\partial v}{\partial r} + \frac{v}{r^2} + \left(\frac{1}{r}\right)\frac{\partial^2 u}{\partial r \partial \theta} - \left(\frac{1}{r^2}\right)\frac{\partial u}{\partial \theta}$$

$$= \frac{\partial^2 v}{\partial r^2} - \frac{1}{r}\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right) + \frac{1}{r}\frac{\partial^2 u}{\partial r \partial \theta}$$

$$\therefore \frac{\partial \gamma_{r\theta}}{\partial r} = \frac{\partial^2 v}{\partial r^2} - \left(\frac{1}{r}\right)\gamma_{r\theta} + \left(\frac{1}{r}\right)\frac{\partial \varepsilon_r}{\partial \theta} \quad (6.9f)$$

Differentiating Equation (6.9e) with respect to r and Equation (6.9f) with respect to θ, we get,

$$\frac{\partial^2 \varepsilon_\theta}{\partial r^2} = \left(\frac{1}{r}\right)\frac{\partial \varepsilon_r}{\partial r} - \left(\frac{1}{r^2}\right)\varepsilon_r + \left(\frac{1}{r}\right)\frac{\partial^3 v}{\partial r^2 \partial \theta} - \left(\frac{1}{r^2}\right)\frac{\partial^2 v}{\partial r \partial \theta} - -\left(\frac{1}{r}\right)\frac{\partial \varepsilon_\theta}{\partial r} + \frac{1}{r^2}\varepsilon_\theta \quad (6.9g)$$

and $\dfrac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} = \dfrac{\partial^3 v}{\partial r^2 \partial \theta} - \left(\dfrac{1}{r}\right)\dfrac{\partial \gamma_{r\theta}}{\partial \theta} + \left(\dfrac{1}{r}\right)\dfrac{\partial^2 \varepsilon_r}{\partial \theta^2}$

or $\left(\dfrac{1}{r}\right)\dfrac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} = \left(\dfrac{1}{r}\right)\dfrac{\partial^3 v}{\partial r^2 \partial \theta} - \left(\dfrac{1}{r^2}\right)\dfrac{\partial \gamma_{r\theta}}{\partial \theta} + \left(\dfrac{1}{r^2}\right)\dfrac{\partial^2 \varepsilon_r}{\partial \theta^2}$ (6.9h)

Subtracting Equation (6.9h) from Equation (6.9g) and using Equation (6.9e), we get,

$$\frac{\partial^2 \varepsilon_\theta}{\partial r^2} - \left(\frac{1}{r}\right)\frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} = \left(\frac{1}{r}\right)\frac{\partial \varepsilon_r}{\partial r} - \left(\frac{\varepsilon_r}{r^2}\right) - \left(\frac{1}{r^2}\right)\frac{\partial^2 v}{\partial r \partial \theta} - \left(\frac{1}{r}\right)\frac{\partial \varepsilon_\theta}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial \gamma_{r\theta}}{\partial \theta} - \left(\frac{1}{r^2}\right)\frac{\partial^2 \varepsilon_r}{\partial \theta^2} + \frac{\varepsilon_\theta}{r^2}$$

$$= \frac{1}{r}\left(\frac{\partial \varepsilon_r}{\partial r}\right) - \frac{1}{r}\left(\frac{\varepsilon_r}{r} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} - \frac{\varepsilon_\theta}{r}\right) - \frac{1}{r}\left(\frac{\partial \varepsilon_\theta}{\partial r} - \frac{1}{r}.\frac{\partial \gamma_{r\theta}}{\partial \theta} + \frac{1}{r}\frac{\partial^2 \varepsilon_r}{\partial \theta^2}\right)$$

$$= \left(\frac{1}{r}\right)\frac{\partial \varepsilon_r}{\partial r} - \left(\frac{1}{r}\right)\frac{\partial \varepsilon_\theta}{\partial r} - \left(\frac{1}{r}\right)\frac{\partial \varepsilon_\theta}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial \gamma_{r\theta}}{\partial \theta} - \left(\frac{1}{r^2}\right)\frac{\partial^2 \varepsilon_r}{\partial \theta^2}$$

$$= \left(\frac{1}{r}\right)\frac{\partial \varepsilon_r}{\partial r} - \left(\frac{2}{r}\right)\frac{\partial \varepsilon_\theta}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial \gamma_{r\theta}}{\partial \theta} - \left(\frac{1}{r^2}\right)\frac{\partial^2 \varepsilon_r}{\partial \theta^2}$$

$$\therefore \left(\frac{1}{r^2}\right)\frac{\partial \gamma_{r\theta}}{\partial \theta} + \left(\frac{1}{r}\right)\frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} = \frac{\partial^2 \varepsilon_\theta}{\partial r^2} + \left(\frac{2}{r}\right)\frac{\partial \varepsilon_\theta}{\partial r} - \left(\frac{1}{r}\right)\frac{\partial \varepsilon_r}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial^2 \varepsilon_r}{\partial \theta^2}$$

Stress-Strain Relations:

In terms of cylindrical coordinates, the stress-strain relations for 3-dimensional state of stress and strain are given by

$$\varepsilon_r = \frac{1}{E}[\sigma_r - \nu(\sigma_\theta + \sigma_z)]$$

$$\varepsilon_\theta = \frac{1}{E}[\sigma_\theta - \nu(\sigma_r + \sigma_z)] \qquad (6.10)$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_r + \sigma_\theta)]$$

For two-dimensional state of stresses and strains, the above equations reduce to,

For Plane Stress Case

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) \qquad (6.11)$$

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta}$$

For Plane Strain Case

$$\varepsilon_r = \frac{(1+\nu)}{E}[(1-\nu)\sigma_r - \gamma\sigma_\theta]$$

$$\varepsilon_\theta = \frac{(1+\nu)}{E}[(1-\nu)\sigma_\theta - \gamma\sigma_r] \qquad (6.12)$$

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta}$$

Airy's Stress function:

With reference to the two-dimensional equations or stress transformation [Equations (2.12a) to (2.12c)], the relationship between the polar stress components σ_r, σ_θ and $\tau_{r\theta}$ and the Cartesian stress components σ_x, σ_y and τ_{xy} can be obtained as below.

$$\begin{aligned}
\sigma_r &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \sin 2\theta \\
\sigma_\theta &= \sigma_y \cos^2\theta + \sigma_x \sin^2\theta - \tau_{xy} \sin 2\theta \\
\tau_{r\theta} &= (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}\cos 2\theta
\end{aligned} \tag{6.13}$$

Now we have,

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} \qquad \sigma_y = \frac{\partial^2\phi}{\partial x^2} \qquad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} \tag{6.14}$$

Substituting (6.14) in (6.13), we get

$$\begin{aligned}
\sigma_r &= \frac{\partial^2\phi}{\partial y^2}\cos^2\theta + \frac{\partial^2\phi}{\partial x^2}\sin^2\theta - \frac{\partial^2\phi}{\partial x\partial y}\sin 2\theta \\
\sigma_\theta &= \frac{\partial^2\phi}{\partial x^2}\cos^2\theta + \frac{\partial^2\phi}{\partial y^2}\sin^2\theta + \frac{\partial^2\phi}{\partial x\partial y}\sin 2\theta \\
\tau_{r\theta} &= \left(\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2}\right)\sin\theta\cos\theta - \frac{\partial^2\phi}{\partial x\partial y}\cos 2\theta
\end{aligned} \tag{6.15}$$

The polar components of stress in terms of Airy's stress functions are as follows.

$$\sigma_r = \left(\frac{1}{r}\right)\frac{\partial\phi}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial^2\phi}{\partial\theta^2} \tag{6.16}$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} \quad \text{and} \quad \tau_{r\theta} = \left(\frac{1}{r^2}\right)\frac{\partial\phi}{\partial\theta} - \left(\frac{1}{r}\right)\frac{\partial^2\phi}{\partial r\partial\theta} \tag{6.17}$$

The above relations can be employed to determine the stress field as a function of r and θ.

AXISYMMETRIC PROBLEMS

Many engineering problems involve solids of revolution subjected to axially symmetric loading. The examples are a circular cylinder loaded by uniform internal or external pressure or other axially symmetric loading, and a semi-infinite half space loaded by a circular area, for example a circular footing on a soil mass. It is convenient to express these problems in terms of the cylindrical co-ordinates. Because of symmetry, the stress components are independent of the angular (q) co-ordinate; hence, all derivatives with respect to q vanish and the components $v, \gamma_{r\theta}, \gamma_{\theta z}, \tau_{r\theta}$ and $\tau_{\theta z}$ are zero.
The non-zero stress components are$\sigma_r, \sigma_\theta,, \sigma_z$ and τ_{rz}.

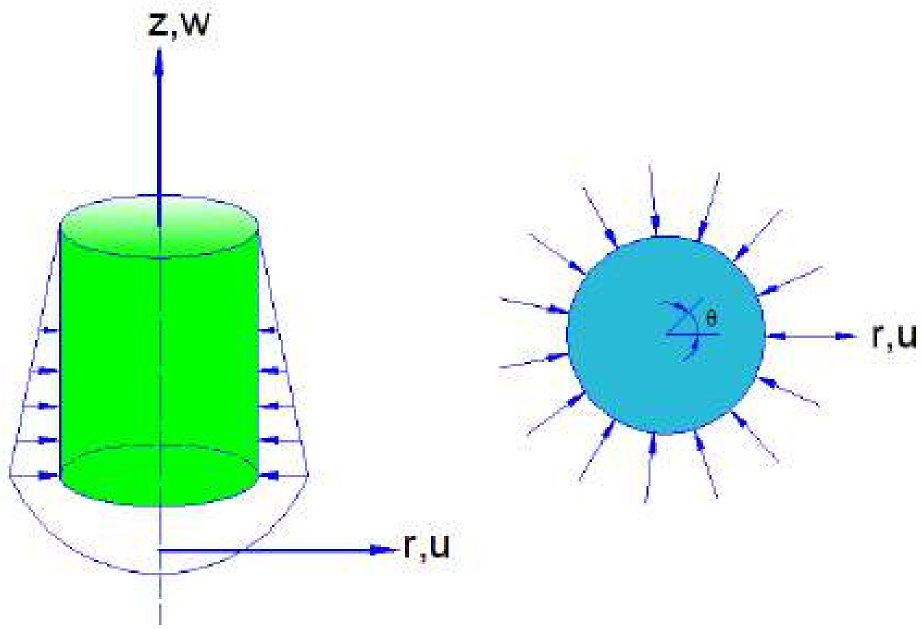
The strain-displacement relations for the non-zero strains become

$$\varepsilon_r = \frac{\partial u}{\partial r}, \varepsilon_\theta = \frac{u}{r}, \varepsilon_z = \frac{\partial w}{\partial z}$$

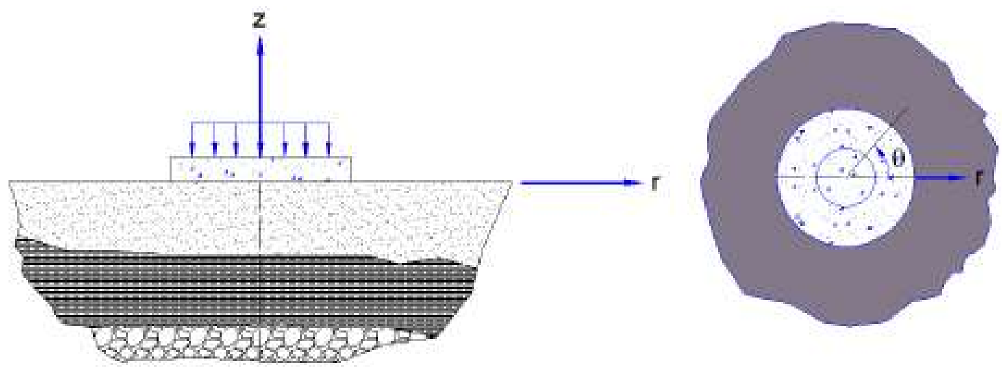
$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

and the constitutive relation is given by

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} (1-v) & v & v & 0 \\ & (1-v) & v & 0 \\ & & (1-v) & 0 \\ & Symmetry & & \frac{(1-2v)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix}$$



(a) Cylinder under axisymmetric loading



(b) Circular Footing on Soil mass

Axisymmetric problems

Thick walled cylinder subjected to internal and external pressure:

Consider a cylinder of inner radius ‘a’ and outer radius ‘b’ as shown in figure below.

Let the cylinder be subjected to internal pressure p_i and an external pressure p_0. This problem can be treated either as a plane stress case ($\sigma_z = 0$) or as a plane strain case ($\varepsilon_z = 0$).

Case (a) Plane Stress

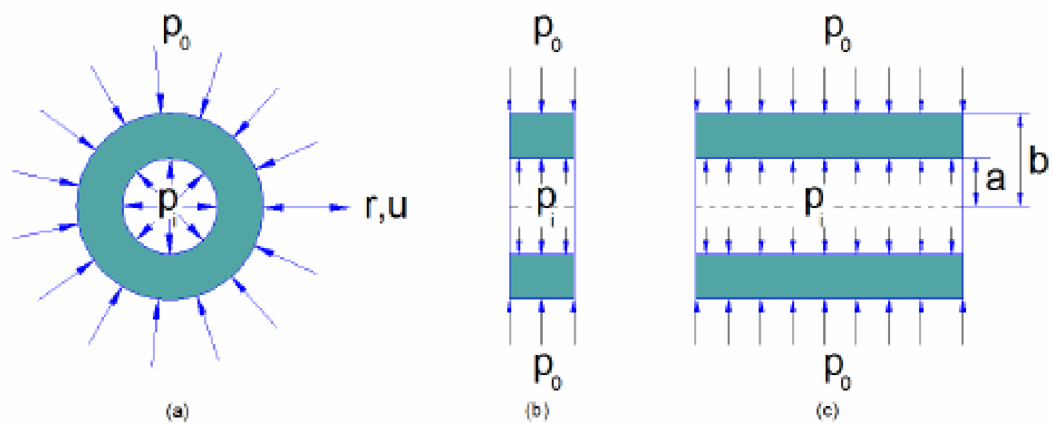


Figure 6.5 (a) Thick-walled cylinder (b) Plane stress case (c) Plane strain case

Consider the ends of the cylinder which are free to expand. Let $\sigma_z = 0$. Owing to uniform radial deformation, $\tau_{rz} = 0$. Neglecting the body forces, equation of equilibrium reduces to

Here σ_θ and σ_r denote the tangential and radial stresses acting normal to the sides of the element.

Since r is the only independent variable, the above equation can be written as

$$\frac{\partial \sigma_r}{\partial r} + \left(\frac{\sigma_r - \sigma_\theta}{r}\right) = 0$$

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta = 0$$

From Hooke's Law,

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta), \quad \varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

Therefore, $\varepsilon_r = \dfrac{du}{dr}$ and $\varepsilon_\theta = \dfrac{u}{r}$ or the stresses in terms of strains are

$$\sigma_r = \frac{E}{(1-\nu^2)}(\varepsilon_r + \nu\varepsilon_\theta)$$

$$\sigma_\theta = \frac{E}{(1-\nu^2)}(\varepsilon_\theta + \nu\varepsilon_r)$$

Substituting the values of ε_r and ε_θ in the above expressions, we get

$$\sigma_r = \frac{E}{(1-\nu^2)}\left(\frac{du}{dr} + \nu\frac{u}{r}\right)$$

$$\sigma_\theta = \frac{E}{(1-\nu^2)}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

Substituting these in the equilibrium Equation (6.21)

$$\frac{d}{dr}\left(r\frac{du}{dr}+\nu u\right)-\left(\frac{u}{r}+\nu\frac{du}{dr}\right)=0$$

$$\frac{du}{dr}+r\frac{d^2u}{dr^2}+\nu\frac{du}{dr}-\frac{u}{r}-\nu\frac{du}{dr}=0$$

$$\text{or}\quad \frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=0$$

The above equation is called equidimensional equation in radial displacement. The solution of the above equation is

$u = C_1 r + C_2/r$

where C_1 and C_2 are constants.

The radial and tangential stresses are written in terms of constants of integration C_1 and C_2.

Therefore,

$$\sigma_r=\frac{E}{(1-\nu^2)}\left[C_1(1+\nu)-C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

$$\sigma_\theta=\frac{E}{(1-\nu^2)}\left[C_1(1+\nu)+C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

The constants are determined from the boundary conditions.

when $r = a,\quad \sigma_r = -p_i$

$r = b,\quad \sigma_r = -p_0$

Hence, $$\frac{E}{(1-\nu^2)}\left[C_1(1+\nu)-C_2\left(\frac{1-\nu}{a^2}\right)\right]=-p_i$$

and $$\frac{E}{(1-\nu^2)}\left[C_1(1+\nu)-C_2\left(\frac{1-\nu}{b^2}\right)\right]=-p_0$$

where the negative sign in the boundary conditions denotes compressive stress.

The constants are evaluated by substitution of equation (6.23a) into (6.23)

$$C_1=\left(\frac{1-\nu}{E}\right)\left(\frac{a^2p_i-b^2p_0}{(b^2-a^2)}\right)$$

$$C_2=\left(\frac{1+\nu}{E}\right)\left(\frac{a^2b^2(p_i-p_0)}{(b^2-a^2)}\right)$$

Substituting the values from earlier equations

$$\sigma_r=\left(\frac{a^2p_i-b^2p_0}{b^2-a^2}\right)-\left(\frac{(p_i-p_0)a^2b^2}{(b^2-a^2)r^2}\right)$$

$$\sigma_\theta = \left(\frac{a^2 p_i - b^2 p_0}{b^2 - a^2} \right) + \left(\frac{(p_i - p_0) a^2 b^2}{(b^2 - a^2) r^2} \right)$$

$$u = \left(\frac{1-\nu}{E} \right) \frac{(a^2 p_i - b^2 p_0) r}{(b^2 - a^2)} + \left(\frac{1+\nu}{E} \right) \frac{(p_i - p_0) a^2 b^2}{(b^2 - a^2) r}$$

These expressions were first derived by G. Lambe.

It is interesting to observe that the sum ($\sigma_r + \sigma_\theta$) is constant through the thickness of the wall of the cylinder, regardless of radial position. Hence according to Hooke's law, the stresses σ_r and σ_θ produce a uniform extension or contraction in z-direction. The cross-sections perpendicular to the axis of the cylinder remain plane. If two adjacent cross-sections are considered, then the deformation undergone by the element does not interfere with the deformation of the neighbouring element. Hence, the elements are considered to be in the plane stress state.

Special Cases

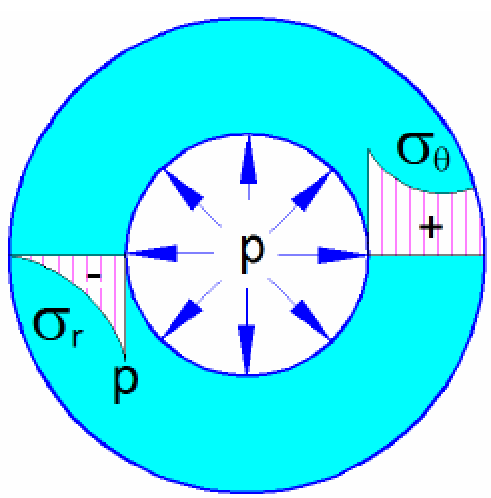
(i) *A cylinder subjected to internal pressure only:* In this case, $p_0 = 0$ and $p_i = p$.

Then Equations (6.24) and (6.25) become

$$\sigma_r = \frac{pa^2}{(b^2 - a^2)} \left(1 - \frac{b^2}{r^2} \right)$$

$$\sigma_\theta = \frac{pa^2}{(b^2 - a^2)} \left(1 + \frac{b^2}{r^2} \right)$$

Figure below shows the variation of radial and circumferential stresses across the thickness of cylinder subjected to internal pressure



Cylinder subjected to internal pressure

The circumferential stress is greater at the inner surface of the cylinder and is given by

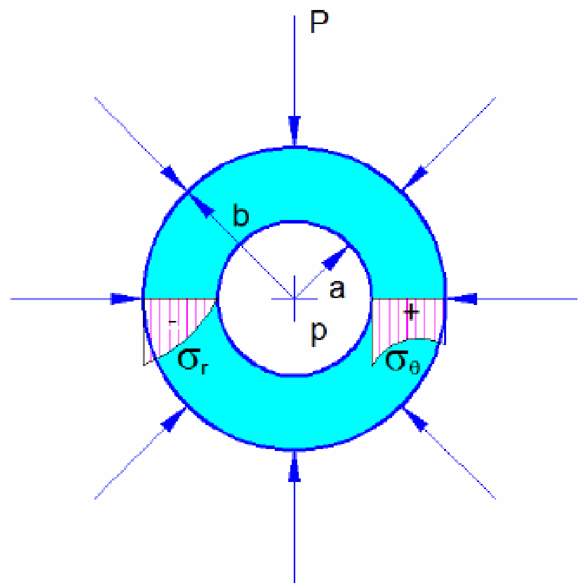
$$(\sigma_\theta)_{max} = \frac{p(a^2 + b^2)}{b^2 - a^2}$$

(ii) *A cylinder subjected to external pressure only:* In this case, $p_i = 0$ and $p_0 = p$.

Equation (6.25) becomes

$$\sigma_r = -\left(\frac{pb^2}{b^2 - a^2}\right)\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_\theta = -\left(\frac{pb^2}{b^2 - a^2}\right)\left(1 + \frac{a^2}{r^2}\right)$$



Cylinder subjected to external pressure

However, if there is no inner hole, i.e., if $a = 0$, the stresses are uniformly distributed in the cylinder as

$$\sigma_r = \sigma_\theta = -p$$

Case (b): Plane Strain

If a long cylinder is considered, sections that are far from the ends are in a state of plane strain and hence σ_z does not vary along the z-axis.

Now, from Hooke's Law,

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \nu(\sigma_\theta + \sigma_z)\right]$$

$$\varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu(\sigma_r + \sigma_z)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r + \sigma_\theta)\right]$$

Since $\varepsilon_z = 0$, then

$$0 = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r + \sigma_\theta)\right]$$

$\sigma_z = \nu\,(\sigma_r + \sigma_\theta)$

Hence,

$$\varepsilon_r = \frac{(1+\nu)}{E}\left[(1-\nu)\sigma_r - \nu\sigma_\theta\right]$$

$$\varepsilon_\theta = \frac{(1+\nu)}{E}\left[(1-\nu)\sigma_\theta - \nu\sigma_r\right]$$

Solving for σ_θ and σ_r,

$$\sigma_\theta = \frac{E}{(1-2\nu)(1+\nu)}\left[\nu\varepsilon_r + (1-\nu)\varepsilon_\theta\right]$$

$$\sigma_r = \frac{E}{(1-2\nu)(1+\nu)}\left[(1-\nu)\varepsilon_r + \nu\varepsilon_\theta\right]$$

Substituting the values of ε_r and ε_θ, the above expressions for σ_θ and σ_r can be written as

$$\sigma_\theta = \frac{E}{(1-2\nu)(1+\nu)}\left[\nu\frac{du}{dr} + (1-\nu)\frac{u}{r}\right]$$

$$\sigma_r = \frac{E}{(1-2\nu)(1+\nu)}\left[(1-\nu)\frac{du}{dr} + \frac{\nu u}{r}\right]$$

Substituting these in the equation of equilibrium (6.21), we get

$$\frac{d}{dr}\left[(1-\nu)r\frac{du}{dr} + \nu u\right] - \nu\frac{du}{dr} - (1-\nu)\frac{u}{r} = 0$$

or $\dfrac{du}{dr} + r\dfrac{d^2u}{dr^2} - \dfrac{u}{r} = 0$

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

The solution of this equation is the same as in Equation (6.22)

$u = C_1 r + C_2/r$

where C_1 and C_2 are constants of integration. Therefore, σ_θ and σ_r are given by

$$\sigma_\theta = \frac{E}{(1-2\nu)(1+\nu)}\left[C_1 + (1-2\nu)\frac{C_2}{r^2}\right]$$

$$\sigma_r = \frac{E}{(1-2\nu)(1+\nu)}\left[C_1 - (1-2\nu)\frac{C_2}{r^2}\right]$$

Applying the boundary conditions,

$\sigma_r = -p_i$ when $r = a$

$\sigma_r = -p_0$ when $r = b$

Therefore, $\frac{E}{(1-2\nu)(1+\nu)}\left[C_1 - (1-2\nu)\frac{C_2}{a^2}\right] = -p_i$

$$\frac{E}{(1-2\nu)(1+\nu)}\left[C_1 - (1-2\nu)\frac{C_2}{b^2}\right] = -p_o$$

Solving, we get

$$C_1 = \frac{(1-2\nu)(1+\nu)}{E}\left(\frac{p_0 b^2 - p_i a^2}{a^2 - b^2}\right)$$

and $\quad C_2 = \frac{(1+\nu)}{E}\left(\frac{(p_0 - p_i)a^2 b^2}{a^2 - b^2}\right)$

Substituting these, the stress components become

$$\sigma_r = \left(\frac{p_i a^2 - p_0 b^2}{b^2 - a^2}\right) - \left(\frac{p_i - p_0}{b^2 - a^2}\right)\frac{a^2 b^2}{r^2}$$

$$\sigma_\theta = \left(\frac{p_i a^2 - p_0 b^2}{b^2 - a^2}\right) + \left(\frac{p_i - p_0}{b^2 - a^2}\right)\frac{a^2 b^2}{r^2}$$

$$\sigma_z = 2\nu\left(\frac{p_0 a^2 - p_i b^2}{b^2 - a^2}\right)$$

It is observed that the values of σ_r *and* σ_θ are identical to those in plane stress case. But in plane stress case, σ_z =0, where as in plane strain case σ_z has a constant value given by above equation.

Rotating Discs of Uniform thickness:

The equation of equilibrium given by

$$\frac{\partial \sigma_r}{\partial r} + \left(\frac{\sigma_r - \sigma_\theta}{r}\right) + F_r = 0 \qquad \text{(a)}$$

is used to treat the case of a rotating disk, provided that the centrifugal "inertia force" is included as a body force. It is assumed that the stresses induced by rotation are distributed symmetrically about the axis of rotation and also independent of disk thickness.Thus, application of equation (a), with the body force per unit volume F_r equated to the centrifugal force $\rho\omega^2 r$, yields

$$\frac{\partial\sigma_r}{\partial r} + \left(\frac{\sigma_r - \sigma_\theta}{r}\right) + \rho\omega^2 r = 0$$

Where ρ is the mass density and ω is the constant angular speed of the disk in rad/sec. the above equation can be written as

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho w^2 r^2 = 0 \quad (6.36)$$

But the strain components are given by

$$\varepsilon_r = \frac{du}{dr} \text{ and } \varepsilon_\theta = \frac{u}{r} \quad (6.37)$$

From Hooke's Law, with $\sigma_z = 0$

$$\varepsilon_r = \frac{1}{E}(\sigma_r - v\sigma_\theta) \quad (6.38)$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - v\sigma_r) \quad (6.39)$$

From equation (6.37),

$$u = r\varepsilon_\theta$$

$$\frac{du}{dr} = \varepsilon_r = \frac{d}{dr}(r\varepsilon_\theta)$$

Using Hooke's Law, we can write equation (6.38) as

$$\frac{1}{E}(\sigma_r - v\sigma_\theta) = \frac{1}{E}\left[\frac{d}{dr}(r\sigma_\theta - vr\sigma_r)\right] \quad (6.40)$$

Let $r\,\sigma_r = y$ (6.41)

Then from equation (6.36)

$$\sigma_\theta = \frac{dy}{dr} + \rho w^2 r^2 \quad (6.42)$$

Substituting these in equation (6.40), we obtain

$$r^2\frac{d^2y}{dr^2} + r\frac{dy}{dr} - y + (3+v) + \rho w^2 r^3 = 0$$

The solution of the above differential equation is

$$y = Cr + C_1\left(\frac{1}{r}\right) - \left(\frac{3+v}{8}\right)\rho w^2 r^3 \quad (6.43)$$

From Equations (6.41) and (6.42), we obtain

$$\sigma_r = C + C_1\left(\frac{1}{r^2}\right) - \left(\frac{3+\nu}{8}\right)\rho w^2 r^2 \qquad (6.44)$$

$$\sigma_\theta = C - C_1\left(\frac{1}{r^2}\right) - \left(\frac{1+3\nu}{8}\right)\rho w^2 r^2 \qquad (6.45)$$

The constants of integration are determined from the boundary conditions.

Solid Disk

For a solid disk, it is required to take $C_1 = 0$, otherwise, the stresses σ_r and σ_θ becomes infinite at the centre. The constant C is determined from the condition at the periphery ($r = b$) of the disk. If there are no forces applied, then

$$(\sigma_r)_{r=b} = C - \left(\frac{3+\nu}{8}\right)\rho w^2 b^2 = 0$$

Therefore, $C = \left(\frac{3+\nu}{8}\right)\rho w^2 b^2$ (6.46)

Hence, Equations (6.44) and (6.45) become,

$$\sigma_r = \left(\frac{3+\nu}{8}\right)\rho w^2 (b^2 - r^2) \qquad (6.47)$$

$$\sigma_\theta = \left(\frac{3+\nu}{8}\right)\rho w^2 b^2 - \left(\frac{1+3\nu}{8}\right)\rho w^2 r^2 \qquad (6.48)$$

The stresses attain their maximum values at the centre of the disk, i.e., at $r = 0$.

Therefore, $\sigma_r = \sigma_{\theta} = \left(\frac{3+\nu}{8}\right)\rho w^2 b^2$ (6.49)

Circular Disk with a hole:

Let a = Radius of the hole.

If there are no forces applied at the boundaries a and b, then

$(\sigma_r)_{r=a} = 0, \quad (\sigma_r)_{r=b} = 0$

from which we find that

$$C = \left(\frac{3+\nu}{8}\right)\rho w^2 (b^2 + a^2)$$

and $C_1 = -\left(\frac{3+\nu}{8}\right)\rho w^2 a^2 b^2$

Substituting the above in Equations (6.44) and (6.45), we obtain

$$\sigma_r = \left(\frac{3+\nu}{8}\right)\rho w^2\left(b^2 + a^2 - \left(\frac{a^2b^2}{r^2}\right) - r^2\right) \tag{6.50}$$

$$\sigma_\theta = \left(\frac{3+\nu}{8}\right)\rho w^2\left(b^2 + a^2 + \left(\frac{a^2b^2}{r^2}\right) - \left(\frac{1+3\nu}{3+\nu}\right)r^2\right) \tag{6.51}$$

The radial stress σ_r reaches its maximum at $r = \sqrt{ab}$, where

$$(\sigma_r)_{max} = \left(\frac{3+\nu}{8}\right)\rho w^2 (b-a)^2 \tag{6.52}$$

The maximum circumferential stress is at the inner boundary, where

$$(\sigma_\theta)_{max} = \left(\frac{3+\nu}{4}\right)\rho w^2\left(b^2 + \left(\frac{1-\nu}{3+\nu}\right)a^2\right) \tag{6.53}$$

The displacement u_r for all the cases considered can be calculated as below:

$$u_r = r\varepsilon_\theta = \frac{r}{E}(\sigma_\theta - \nu\sigma_r) \tag{6.54}$$

Stress concentration

While discussing the case of simple tension and compression, it has been assumed that the bar has a prismatical form. Then for centrally applied forces, the stress at some distance from the ends is uniformly distributed over the cross-section. Abrupt changes in cross-section give rise to great irregularities in stress distribution. These irregularities are of particular importance in the design of machine parts subjected to variable external forces and to reversal of stresses. If there exists in the structural or machine element a discontinuity that interrupts the stress path, the stress at that discontinuity may be considerably greater than the nominal stress on the section; thus there is a “Stress Concentration” at the discontinuity.
The ratio of the maximum stress to the nominal stress on the section is known as the 'Stress Concentration Factor'. Thus, the expression for the maximum normal stress in a centrically loaded member becomes

$$\sigma = K\left(\frac{P}{A}\right) \tag{6.55}$$

where *A* is either gross or net area (area at the reduced section), *K* = stress concentration factor and *P* is the applied load on the member. In Figures 6.8 (a), 6.8(b) and 6.8(c), the type of discontinuity is shown and in Figures

6.8(d), 6.8(e) and 6.8(f) the approximate distribution of normal stress on a transverse plane is shown.

Stress concentration is a matter, which is frequently overlooked by designers. The high stress concentration found at the edge of a hole is of great practical importance. As an example, holes in ships decks may be mentioned. When the hull of a ship is bent, tension or compression is produced in the decks and there is a high stress concentration at the holes. Under the cycles of stress produced by waves, fatigue of the metal at the overstressed portions may result finally in fatigue cracks.

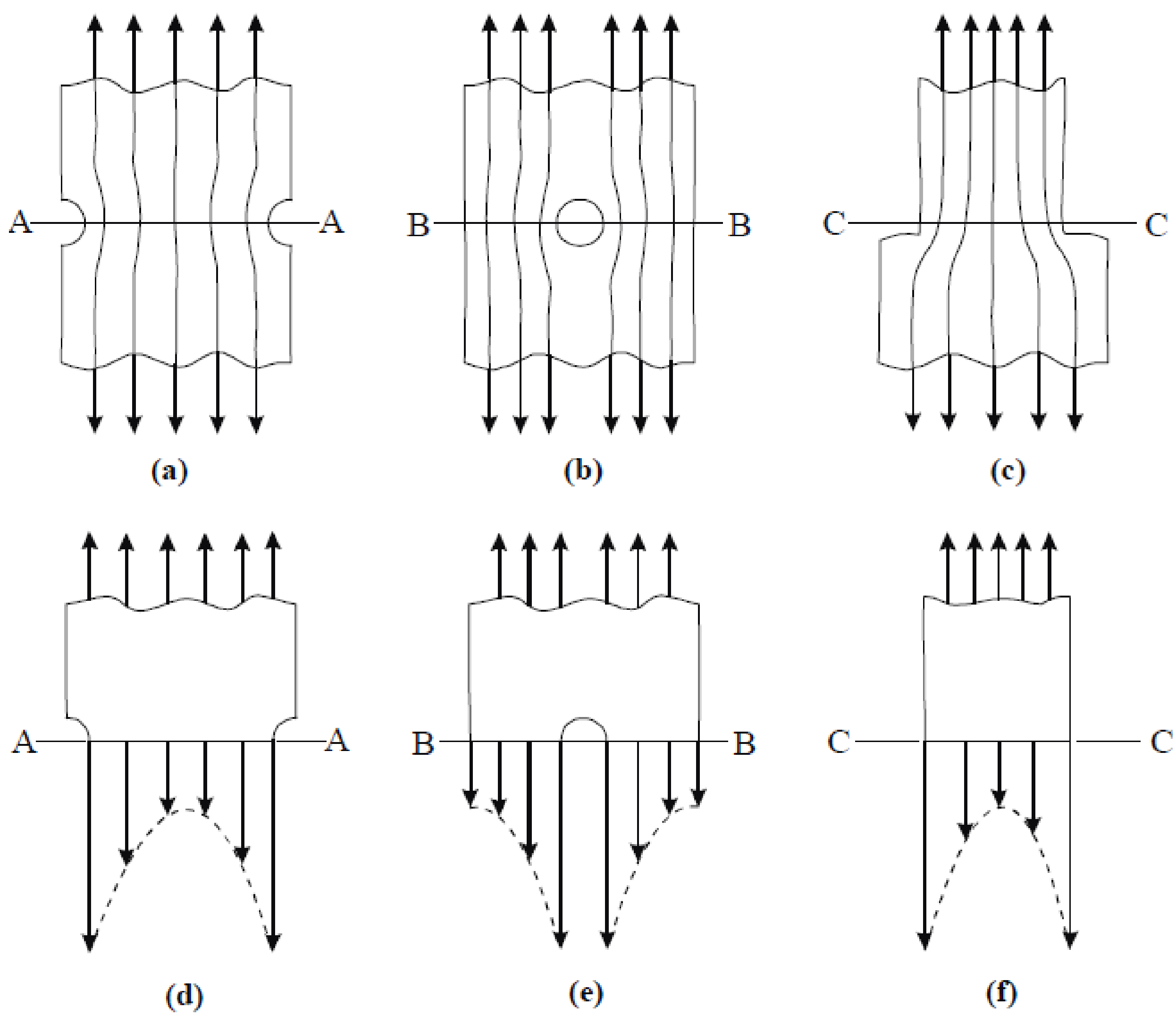


Figure 6.8 Irregularities in Stress distribution

Bending of beams and plates:

Bending of a cantilever Beam: Consider a cantilever having a narrow rectangular cross section of unit width bent by a force P applied at the end (Fig. 26). The upper and lower edges are free from load, and shearing forces, having a resultant P, are distributed along the end $x = 0$. These conditions can be satisfied by a proper combination of pure shear, with the stresses (e) of Art. 17 represented in Fig. 24. Superposing the pure shear $\tau_{xy} = -b_2$ on the stresses (e), we find

$$\sigma_x = d_4 xy, \qquad \sigma_y = 0$$
$$\tau_{xy} = -b_2 - \frac{d_4}{2} y^2 \qquad (a)$$

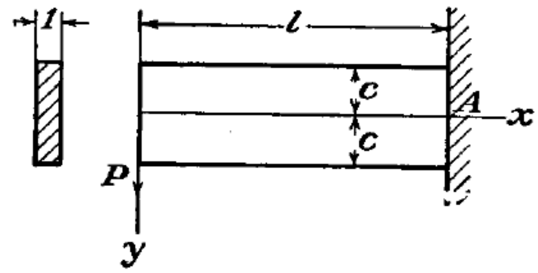


FIG. 26.

To have the longitudinal sides $y = \pm c$ free from forces we must have

$$(\tau_{xy})_{y=\pm c} = -b_2 - \frac{d_4}{2} c^2 = 0$$

from which

$$d_4 = -\frac{2b_2}{c^2}$$

To satisfy the condition on the loaded end the sum of the shearing forces distributed over this end must be equal to P. Hence[1]

$$-\int_{-c}^{c} \tau_{xy} \cdot dy = \int_{-c}^{c} \left(b_2 - \frac{b_2}{c^2} y^2\right) dy = P$$

from which

$$b_2 = \frac{3}{4}\frac{P}{c}$$

Substituting these values of d_4 and b_2 in Eqs. (a) we find

$$\sigma_x = -\frac{3}{2}\frac{P}{c^3} xy, \qquad \sigma_y = 0$$
$$\tau_{xy} = -\frac{3P}{4c}\left(1 - \frac{y^2}{c^2}\right)$$

Noting that $\frac{2}{3}c^3$ is the moment of inertia I of the cross section of the cantilever, we have

$$\sigma_x = -\frac{Pxy}{I}, \qquad \sigma_y = 0$$
$$\tau_{xy} = -\frac{P}{I}\frac{1}{2}(c^2 - y^2) \tag{b}$$

This coincides completely with the elementary solution as given in books on the strength of materials. It should be noted that this solution represents an exact solution only if the shearing forces on the ends are distributed according to the same parabolic law as the shearing stress τ_{xy} and the intensity of the normal forces at the built-in end is proportional to y. If the forces at the ends are distributed in any other manner, the stress distribution (*b*) is not a correct solution for the ends of the cantilever, but, by virtue of Saint-Venant's principle, it can be considered satisfactory for cross sections at a considerable distance from the ends.

Let us consider now the displacement corresponding to the stresses (*b*). Applying Hooke's law we find

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\sigma_x}{E} = -\frac{Pxy}{EI}, \qquad \epsilon_y = \frac{\partial v}{\partial y} = -\frac{\nu\sigma_x}{E} = \frac{\nu Pxy}{EI} \tag{c}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{G} = -\frac{P}{2IG}(c^2 - y^2) \tag{d}$$

The procedure for obtaining the components u and v of the displacement consists in integrating Eqs. (*c*) and (*d*). By integration of Eqs. (*c*) we find

$$u = -\frac{Px^2y}{2EI} + f(y), \qquad v = \frac{\nu Pxy^2}{2EI} + f_1(x)$$

in which $f(y)$ and $f_1(x)$ are as yet unknown functions of y only and x only. Substituting these values of u and v in Eq. (*d*) we find

$$-\frac{Px^2}{2EI} + \frac{df(y)}{dy} + \frac{\nu Py^2}{2EI} + \frac{df_1(x)}{dx} = -\frac{P}{2IG}(c^2 - y^2)$$

In this equation some terms are functions of x only, some are functions of y only, and one is independent of both x and y. Denoting these groups by $F(x)$, $G(y)$, K, we have

$$F(x) = -\frac{Px^2}{2EI} + \frac{df_1(x)}{dx}, \qquad G(y) = \frac{df(y)}{dy} + \frac{\nu Py^2}{2EI} - \frac{Py^2}{2IG}$$

$$K = -\frac{Pc^2}{2IG}$$

and the equation may be written

$$F(x) + G(y) = K$$

Such an equation means that $F(x)$ must be some *constant* d and $G(y)$ some constant e. Otherwise $F(x)$ and $G(y)$ would vary with x and y, respectively, and by varying x alone, or y alone, the equality would be violated. Thus

$$e + d = -\frac{Pc^2}{2IG} \tag{e}$$

and

$$\frac{df_1(x)}{dx} = \frac{Px^2}{2EI} + d, \qquad \frac{df(y)}{dy} = -\frac{Py^2}{2EI} + \frac{Py^2}{2IG} + e$$

Functions $f(y)$ and $f_1(x)$ are then

$$f(y) = -\frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + ey + g$$

$$f_1(x) = \frac{Px^3}{6EI} + dx + h$$

Substituting in the expressions for u and v we find

$$u = -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + ey + g$$

$$v = \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} + dx + h \tag{g}$$

The constants d, e, g, h may now be determined from Eq. (e) and from the three conditions of constraint which are necessary to prevent the beam from moving as a rigid body in the xy-plane. Assume that the point A, the centroid of the end cross section, is fixed. Then u and v are zero for $x = l$, $y = 0$, and we find from Eqs. (g),

$$g = 0, \qquad h = -\frac{Pl^3}{6EI} - dl$$

The deflection curve is obtained by substituting $y = 0$ into the second of Eqs. (g). Then

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^3}{6EI} - d(l - x) \tag{h}$$

For determining the constant d in this equation we must use the third condition of constraint, eliminating the possibility of rotation of the beam in the xy-plane about the fixed point A. This constraint can be realized in various ways. Let us consider two cases: (1) When an element of the axis of the beam is fixed at the end A. Then the condition of constraint is

$$\left(\frac{\partial v}{\partial x}\right)_{\substack{x=l\\y=0}} = 0 \tag{k}$$

(2) When a vertical element of the cross section at the point A is fixed. Then the condition of constraint is

$$\left(\frac{\partial u}{\partial y}\right)_{\substack{x=l\\y=0}} = 0 \tag{l}$$

In the first case we obtain from Eq. (h)

$$d = -\frac{Pl^2}{2EI}$$

and from Eq. (e) we find

$$e = \frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}$$

Substituting all the constants in Eqs. (g), we find

$$\begin{aligned} u &= -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + \left(\frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}\right)y \\ v &= \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} \end{aligned} \tag{m}$$

The equation of the deflection curve is

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} \tag{n}$$

which gives for the deflection at the loaded end ($x = 0$) the value $Pl^3/3EI$. This coincides with the value usually derived in elementary books on the strength of materials.

To illustrate the distortion of cross sections produced by shearing stresses let us consider the displacement u at the fixed end ($x = l$). For this end we have from Eqs. (m),

$$\begin{aligned} (u)_{x=l} &= -\frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} - \frac{Pc^2y}{2IG} \\ \left(\frac{\partial u}{\partial y}\right)_{x=l} &= -\frac{\nu Py^2}{2EI} + \frac{Py^2}{2IG} - \frac{Pc^2}{2IG} \\ \left(\frac{\partial u}{\partial y}\right)_{\substack{x=l\\y=0}} &= -\frac{Pc^2}{2IG} = -\frac{3}{4}\frac{P}{cG} \end{aligned} \tag{o}$$

The shape of the cross section after distortion is as shown in Fig. 27a. Due to the shearing stress $\tau_{xy} = -3P/4c$ at the point A, an element of the cross section at A rotates in the xy-plane about the point A through an angle $3P/4cG$ in the clockwise direction.

If a vertical element of the cross section is fixed at A (Fig. 27b), instead of a horizontal element of the axis, we find from condition (l) and the first of Eqs. (g)

$$e = \frac{Pl^2}{2EI}$$

and from Eq. (e) we find

$$d = -\frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}$$

Substituting in the second of Eqs. (g) we find

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} + \frac{Pc^2}{2IG}(l - x) \tag{r}$$

Comparing this with Eq. (n) it can be concluded that, due to rotation

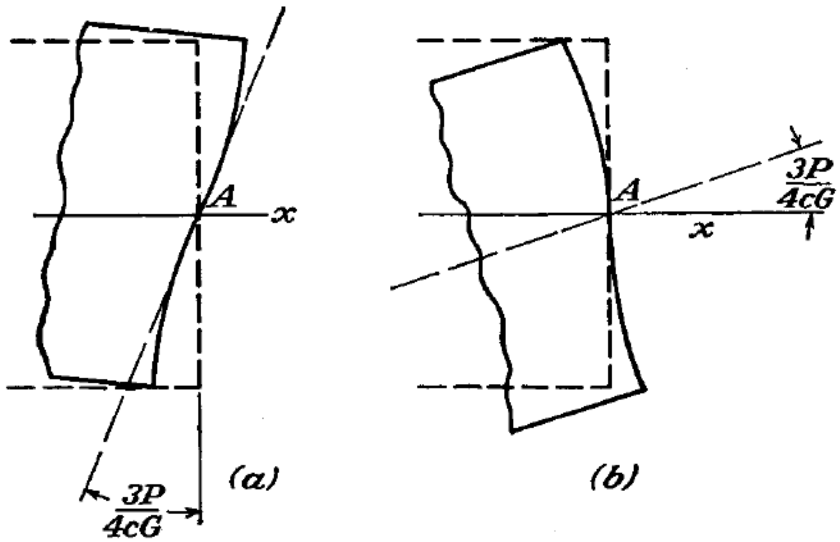


FIG. 27.

of the end of the axis at A (Fig. 27b), the deflections of the axis of the cantilever are increased by the quantity

$$\frac{Pc^2}{2IG}(l - x) = \frac{3P}{4cG}(l - x)$$

This is the so-called *effect of shearing force* on the deflection of the beam. In practice, at the built-in end we have conditions different from those shown in Fig. 27. The fixed section is usually not free to distort and the distribution of forces at this end is different from that given by Eqs. (b). Solution (b) is, however, satisfactory for comparatively long cantilevers at considerable distances from the terminals.

Bending of beam by uniform loading: Let a beam of narrow rectangular cross section of unit width, supported at the ends, be bent

by a uniformly distributed load of intensity q, as shown in Fig. 28 The conditions at the upper and lower edges of the beam are:

$$(\tau_{xy})_{y=\pm c} = 0, \qquad (\sigma_y)_{y=+c} = 0, \qquad (\sigma_y)_{y=-c} = -q \tag{a}$$

The conditions at the ends $x = \pm l$ are

$$\int_{-c}^{c} \tau_{xy}\, dy = \mp ql, \qquad \int_{-c}^{c} \sigma_x\, dy = 0, \qquad \int_{-c}^{c} \sigma_x y\, dy = 0 \tag{b}$$

The last two of Eqs. (b) state that there is no longitudinal force and no bending couple applied at the ends of the beam. All the conditions (a) and (b) can be satisfied by combining certain solutions in the form

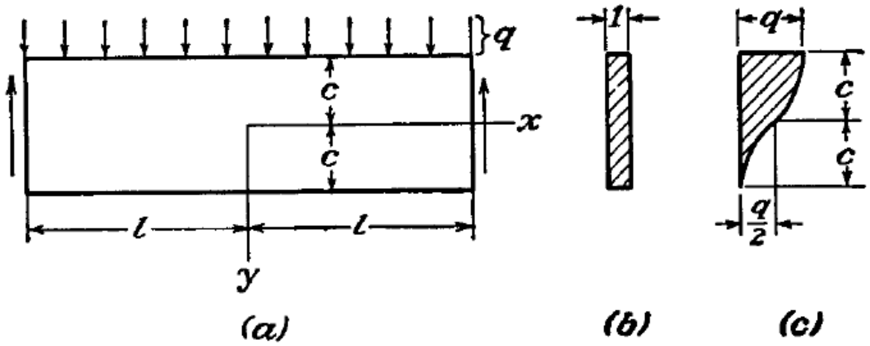


FIG. 28.

of polynomials as obtained in Art. 17. We begin with solution (g) illustrated by Fig. 25. To remove the tensile stresses along the side $y = c$ and the shearing stresses along the sides $y = \pm c$ we superpose a simple compression $\sigma_y = a_2$ from solution (b), Art. 17, and the stresses $\sigma_y = b_3 y$ and $\tau_{xy} = -b_3 x$ in Fig. 23. In this manner we find

$$\begin{aligned} \sigma_x &= d_5(x^2y - \tfrac{2}{3}y^3) \\ \sigma_y &= \tfrac{1}{3}d_5y^3 + b_3y + a_2 \\ \tau_{xy} &= -d_5xy^2 - b_3x \end{aligned} \qquad (c)$$

From the conditions (a) we find

$$\begin{aligned} -d_5c^2 - b_3 &= 0 \\ \tfrac{1}{3}d_5c^3 + b_3c + a_2 &= 0 \\ -\tfrac{1}{3}d_5c^3 - b_3c + a_2 &= -q \end{aligned}$$

from which

$$a_2 = -\frac{q}{2}, \qquad b_3 = \frac{3}{4}\frac{q}{c}, \qquad d_5 = -\frac{3}{4}\frac{q}{c^3}$$

Substituting in Eqs. (c) and noting that $2c^3/3$ is equal to the moment of inertia I of the rectangular cross-sectional area of unit width, we find

$$\begin{aligned} \sigma_x &= -\frac{3}{4}\frac{q}{c^3}\left(x^2y - \frac{2}{3}y^3\right) = -\frac{q}{2I}\left(x^2y - \frac{2}{3}y^3\right) \\ \sigma_y &= -\frac{3q}{4c^3}\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right) = -\frac{q}{2I}\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right) \end{aligned} \qquad (d)$$

$$\tau_{xy} = -\frac{3q}{4c^3}(c^2 - y^2)x = -\frac{q}{2I}(c^2 - y^2)x$$

It can easily be checked that these stress components satisfy not only conditions (a) on the longitudinal sides but also the first two conditions (b) at the ends. To make the couples at the ends of the beam vanish we superpose on solution (d) a pure bending, $\sigma_x = d_3 y$, $\sigma_y = \tau_{xy} = 0$, shown in Fig. 22, and determine the constant d_3 from the condition at $x = \pm l$

$$\int_{-c}^{c} \sigma_x y \, dy = \int_{-c}^{c} \left[-\frac{3}{4}\frac{q}{c^3}\left(l^2 y - \frac{2}{3} y^3\right) + d_3 y \right] y \, dy = 0$$

from which

$$d_3 = \frac{3}{4}\frac{q}{c}\left(\frac{l^2}{c^2} - \frac{2}{5}\right)$$

Hence, finally,

$$\begin{aligned} \sigma_x &= -\frac{3}{4}\frac{q}{c^3}\left(x^2 y - \frac{2}{3} y^3\right) + \frac{3}{4}\frac{q}{c}\left(\frac{l^2}{c^2} - \frac{2}{5}\right) y \\ &= \frac{q}{2I}(l^2 - x^2)y + \frac{q}{2I}\left(\frac{2}{3} y^3 - \frac{2}{5} c^2 y\right) \end{aligned} \tag{33}$$

The first term in this expression represents the stresses given by the usual elementary theory of bending, and the second term gives the necessary correction. This correction does not depend on x and is small in comparison with the maximum bending stress, provided the span of the beam is large in comparison with its depth. For such beams the elementary theory of bending gives a sufficiently accurate value for the stresses σ_x. It should be noted that expression (33) is an exact solution only if at the ends $x = \pm l$ the normal forces are distributed according to the law

$$\bar{X} = \frac{3}{4}\frac{q}{c^3}\left(\frac{2}{3} y^3 - \frac{2}{5} c^2 y\right)$$

i.e., if the normal forces at the ends are the same as σ_x for $x = \pm l$ from Eq. (33). These forces have a resultant force and a resultant couple equal to zero. Hence, from Saint-Venant's principle we can conclude that their effects on the stresses at considerable distances from the ends, say at distances larger than the depth of the beam, can be neglected. Solution (33) at such points is therefore accurate enough for the case when there are no forces $\bar{X}$.

The discrepancy between the exact solution (33) and the approximate solution, given by the first term of (33), is due to the fact that in deriving the approximate solution it is assumed that the longitudinal fibers of the beam are in a condition of simple tension. From solution (*d*) it can be seen that there are compressive stresses σ_y between the fibers. These stresses are responsible for the correction represented by the second term of solution (33). The distribution of the compressive stresses σ_y over the depth of the beam is shown in Fig. 28*c*. The distribution of shearing stress τ_{xy}, given by the third of Eqs. (*d*), over a cross section of the beam coincides with that given by the usual elementary theory.

When the beam is loaded by its own weight instead of the distributed load q, the solution must be modified by putting $q = 2\rho gc$ in (33) and the last two of Eqs. (*d*), and adding the stresses

$$\sigma_x = 0, \qquad \sigma_y = \rho g(c - y), \qquad \tau_{xy} = 0 \tag{e}$$

For the stress distribution (*e*) can be obtained from Eqs. (29) by taking

$$\phi = \tfrac{1}{2}\rho g(cx^2 + y^3/3)$$

and therefore represents a possible state of stress due to weight and boundary forces. On the upper edge $y = -c$ we have $\sigma_y = 2\rho gc$, and on the lower edge $y = c$, $\sigma_y = 0$. Thus when the stresses (*e*) are added to the previous solution, with $q = 2\rho gc$, the stress on both horizontal edges is zero, and the load on the beam consists only of its own weight.

The displacements u and v can be calculated by the method indicated in the previous article. Assuming that at the centroid of the middle cross section ($x = 0$, $y = 0$) the horizontal displacement is zero and the vertical displacement is equal to the deflection δ, we find, using solutions (*d*) and (33),

$$u = \frac{q}{2EI}\left[\left(l^2x - \frac{x^3}{3}\right)y + x\left(\frac{2}{3}y^3 - \frac{2}{5}c^2y\right) + \nu x\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right)\right]$$

$$v = -\frac{q}{2EI}\left\{\frac{y^4}{12} - \frac{c^2y^2}{2} + \frac{2}{3}c^3y + \nu\left[(l^2 - x^2)\frac{y^2}{2} + \frac{y^4}{6} - \frac{1}{5}c^2y^2\right]\right\}$$
$$- \frac{q}{2EI}\left[\frac{l^2x^2}{2} - \frac{x^4}{12} - \frac{1}{5}c^2x^2 + \left(1 + \frac{1}{2}\nu\right)c^2x^2\right] + \delta$$

It can be seen from the expression for u that the neutral surface of the beam is not at the center line. Due to the compressive stress

$$(\sigma_y)_{y=0} = -\frac{q}{2}$$

the center line has a tensile strain $\nu q/2E$, and we find

$$(u)_{y=0} = \frac{\nu q x}{2E}$$

From the expression for v we find the equation of the deflection curve,

$$(v)_{y=0} = \delta - \frac{q}{2EI}\left[\frac{l^2x^2}{2} - \frac{x^4}{12} - \frac{1}{5}c^2x^2 + \left(1 + \frac{1}{2}\nu\right)c^2x^2\right] \qquad (f)$$

Assuming that the deflection is zero at the ends ($x = \pm l$) of the center line, we find

$$\delta = \frac{5}{24}\frac{ql^4}{EI}\left[1 + \frac{12}{5}\frac{c^2}{l^2}\left(\frac{4}{5} + \frac{\nu}{2}\right)\right] \qquad (34)$$

The factor before the brackets is the deflection which is derived by the elementary analysis, assuming that cross sections of the beam remain plane during bending. The second term in the brackets represents the correction usually called the *effect of shearing force.*

By differentiating Eq. (f) for the deflection curve twice with respect to x, we find the following expression for the curvature:

$$\left(\frac{d^2v}{dx^2}\right)_{y=0} = \frac{q}{EI}\left[\frac{l^2 - x^2}{2} + c^2\left(\frac{4}{5} + \frac{\nu}{2}\right)\right] \qquad (35)$$

It will be seen that the curvature is not exactly proportional to the bending moment[1] $q(l^2 - x^2)/2$. The additional term in the brackets represents the necessary correction to the usual elementary formula.

Kirkhhof and Mindlin concept

Module-III.

Torsion of Prismatic Bars (St.Venant's approach)

From the study of elementary strength of materials, two important expressions related to the torsion of circular bars were developed. They are

$$\tau = \frac{M_t r}{J}$$

and $$\theta = \frac{1}{L}\int_L \frac{M_t dz}{GJ}$$

Here τ represents the shear stress, M_t the applied torque, r the radius at which the stress is required, G the shear modulus, θ the angle of twist per unit longitudinal length, L the length, and z the axial co-ordinate.

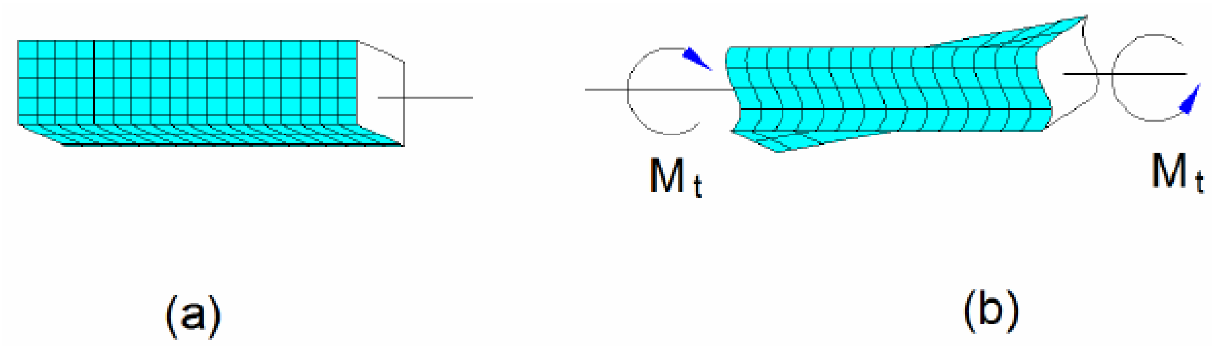
Also, J = Polar moment of inertia which is defined by $\int_A r^2 dA$

The following are the assumptions associated with the elementary approach in deriving above equations.

1. The material is homogeneous and obeys Hooke's Law.
2. All plane sections perpendicular to the longitudinal axis remain plane following the application of a torque, i.e., points in a given cross-sectional plane remain in that plane after twisting.
3. Subsequent to twisting, cross-sections are undistorted in their individual planes, i.e., the shearing strain varies linearly with the distance from the central axis.
4. Angle of twist per unit length is constant.

In most cases, the members that transmit torque, such as propeller shaft and torque tubes of power equipment, are circular or turbular in cross-section.

But in some cases, slender members with other than circular cross-sections are used. These are shown in the Figure 7.0.



Non-Circular Sections Subjected to Torque

While treating non-circular prismatic bars, initially plane cross-sections [Figure (a)] experience out-of-plane deformation or "Warping" [Figure (b)] and therefore assumptions 2. and 3. are no longer appropriate. Consequently, a different analytical approach is employed, using theory of elasticity.

Genera Solution of the Torsion Problem:

The correct solution of the problem of torsion of bars by couples applied at the ends was given by **Saint-Venant**. He used the semi-inverse method. In the beginning, he made certain assumptions for the deformation of the twisted bar and showed that these assumptions could satisfy the equations of equilibrium given by

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + F_z = 0$$

and the boundary conditions such as

$$\overline{X} = \sigma_x l + \tau_{xy} m + \tau_{xz} n$$

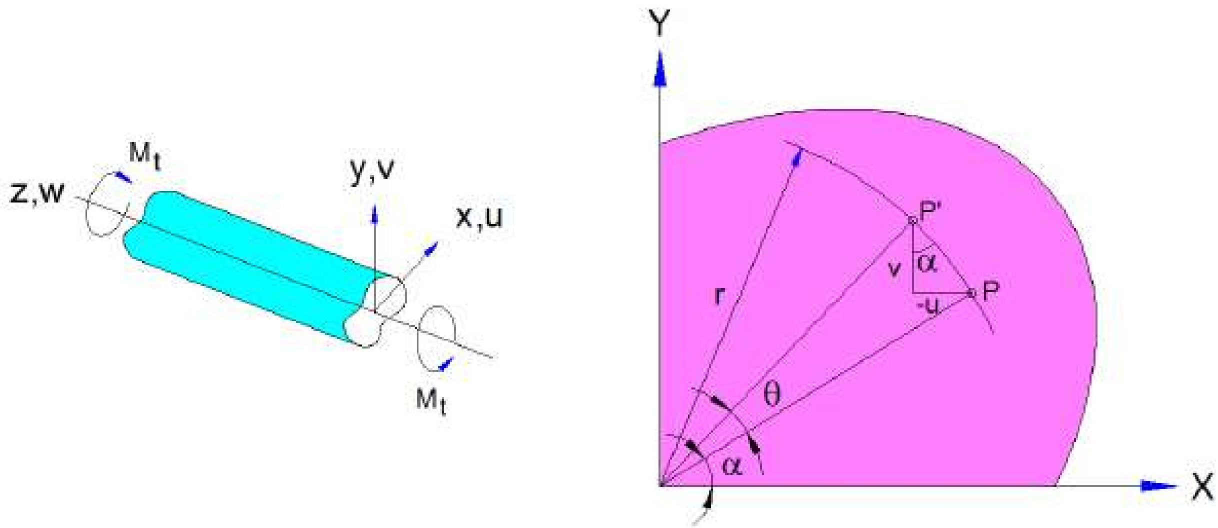
$$\overline{Y} = \sigma_y m + \tau_{yz} m + \tau_{xy} l$$

$$\overline{Z} = \sigma_z n + \tau_{xz} l + \tau_{yz} m$$

in which F_x, F_y, F_z are the body forces, X, Y, Z are the components of the surface forces per unit area and l, m, n are the direction cosines.

Also from the uniqueness of solutions of the elasticity equations, it follows that the torques on the ends are applied as shear stress in exactly the manner required by the solution itself.

Now, consider a prismatic bar of constant arbitrary cross-section subjected to equal and opposite twisting moments applied at the ends, as shown in the Figure below.



Bars subjected to torsion

Saint-Venant assumes that the deformation of the twisted shaft consists of

1. Rotations of cross-sections of the shaft as in the case of a circular shaft and
2. Warping of the cross-sections that is the same for all cross-sections.

The origin of x, y, z in the figure is located at the center of the twist of the cross-section, about which the cross-section rotates during twisting. Figure 7.1(b) shows the partial end view of the bar (and could represent any section). An arbitrary point on the cross-section, point $P(x, y)$, located a distance r from center of twist A, has moved to $P'(x\text{-}u,\ y\text{+}v)$ as a result of torsion. Assuming that no rotation occurs at end $z = 0$ and that θ is small, the x and y displacements of P are respectively:

$u = -(r\theta_z)\sin\alpha$

But $\sin\alpha = y / r$

Therefore, $u = -(r\theta_z)\, y/r = -y\theta_z$ (a)

Similarly, $v = (r\theta_z)\cos\alpha = (r\theta_z)\, x/r = x\theta_z$ (b)

where θ_z is the angle of rotation of the cross-section at a distance z from the origin.

The warping of cross-sections is defined by a function ψ as

$w = \theta\, \psi(x, y)$ (c)

Here, the equations (a) and (b) specify the rigid body rotation of any cross-section through a small angle θ_z. However, with the assumed displacements (a), (b) and (c), we calculate the components of strain from the equations given below.

$$\varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_y = \frac{\partial v}{\partial y},\ \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

and $$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z},$$

Substituting (a), (b) and (c) in the above equations, we obtain

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} - y\,\theta = \left(\theta \frac{\partial \psi}{\partial x} - y\theta\right)$$

or $$\gamma_{xz} = \theta\left(\frac{\partial \psi}{\partial x} - y\right)$$

and $$\gamma_{yz} = \frac{\partial w}{\partial y} + x\,\theta = \left(\theta \frac{\partial \psi}{\partial y} + x\theta\right)$$

or $$\gamma_{yz} = \theta\left(\frac{\partial \psi}{\partial y} + x\right)$$

Also, by Hooke's Law, the stress-strain relationships are given by

$$\sigma_x = 2G\varepsilon_x + \lambda e,\quad \tau_{xy} = G\gamma_{xy}$$

$$\sigma_y = 2G\varepsilon_y + \lambda e,\quad \tau_{yz} = G\gamma_{yz}$$

$$\sigma_z = 2G\varepsilon_z + \lambda e,\quad \tau_{xz} = \gamma_{xz}$$

where $e = \varepsilon_x + \varepsilon_y + \varepsilon_z$

and $$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

Substituting (a), (b) and (c) in the above equations, we obtain

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

$$\tau_{xz} = G\left(\frac{\partial w}{\partial x} - y\theta\right) = G\theta\left(\frac{\partial \psi}{\partial x} - y\right) \quad \text{(d)}$$

$$\tau_{yz} = G\left(\frac{\partial w}{\partial y} + x\theta\right) = G\theta\left(\frac{\partial \psi}{\partial y} + x\right) \quad \text{(e)}$$

It can be observed that with the assumptions (a), (b) and (c) regarding deformation, there will be no normal stresses acting between the longitudinal fibers of the shaft or in the longitudinal direction of those fibers. Also, there will be no distortion in the planes of cross-sections, since ε_x, ε_y and γ_{xy} vanish. We have at each point, pure shear defined by the components τ_{xz} and τ_{yz}.

However, the stress components should satisfy the equations of equilibrium given by:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0 \qquad \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + F_z = 0$$

Assuming negligible body forces, and substituting the stress components into equilibrium equations, we obtain

$$\frac{\partial \tau_{xz}}{\partial z} = 0, \qquad \frac{\partial \tau_{zy}}{\partial z} = 0, \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0$$

Also, the function $\psi\,(x, y)$, defining warping of cross-section must be determined by the equations of equilibrium.

Therefore, we find that the function ψ must satisfy the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Now, differentiating equation (d) with respect to y and the equation (e) with respect to x, and subtracting we get an equation of compatibility

Hence, $\frac{\partial \tau_{xz}}{\partial y} = -G\theta$

$$\frac{\partial \tau_{yz}}{\partial x} = G\theta$$

$$\frac{\partial \tau_{xz}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x} = -G\theta - G\theta = -2G\theta = H$$

Therefore, $\frac{\partial \tau_{xz}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x} = H$

Therefore the stress in a bar of arbitrary section may be determined by solving above Equations along with the given boundary conditions.

Boundary Conditions:

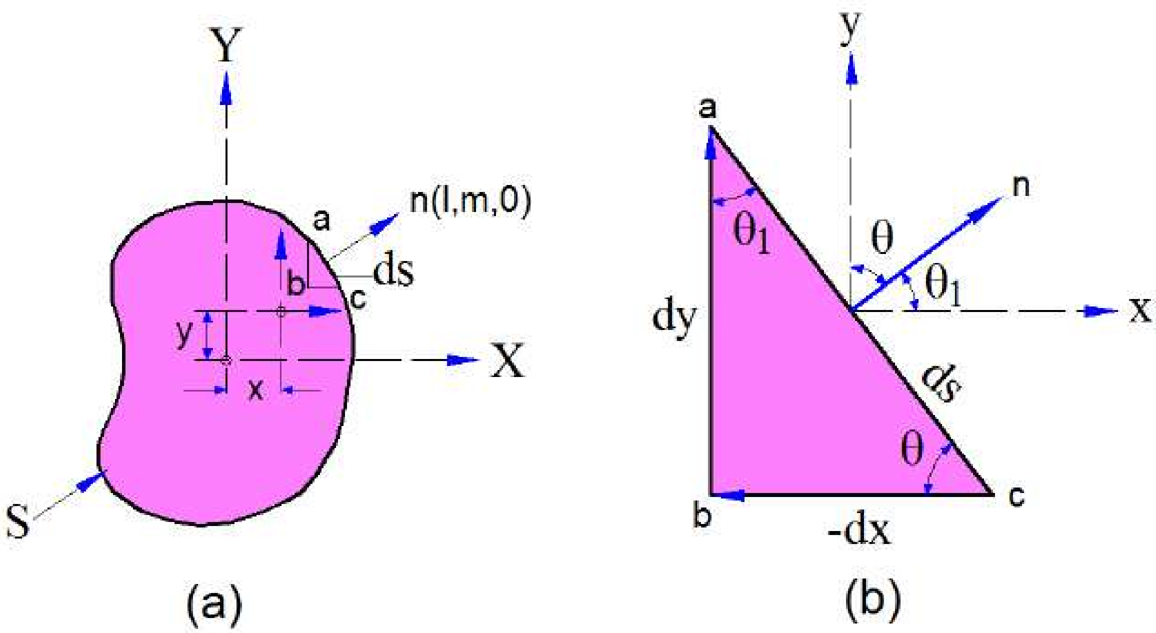
Now, consider the boundary conditions given by

$$\overline{X} = \sigma_x l + \tau_{xy} m + \tau_{xz} n$$
$$\overline{Y} = \sigma_y m + \tau_{yz} n + \tau_{xy} l$$
$$\overline{Z} = \sigma_z n + \tau_{xz} l + \tau_{yz} m$$

For the lateral surface of the bar, which is free from external forces acting on the boundary and the normal n to the surface is perpendicular to the z-axis, we have $X = Y = Z = 0$ and $n = 0$. The first two equations are identically satisfied and the third gives,

$$\tau_{xz} l + \tau_{yz} m = 0$$

which means that the resultant shearing stress at the boundary is directed along the tangent to the boundary, as shown in the Figure below.



Cross-section of the bar & Boundary conditions

Considering an infinitesimal element abc at the boundary and assuming that S is increasing in the direction from c to a,

$$l = \cos(N, x) = \frac{dy}{dS}$$

$$m = \cos(N, y) = -\frac{dx}{dS}$$

The above equation becomes

$$\tau_{xz}\left(\frac{dy}{dS}\right)-\tau_{yz}\left(\frac{dx}{dS}\right)=0$$

or $$\left(\frac{\partial\psi}{\partial x}-y\right)\left(\frac{dy}{dS}\right)-\left(\frac{\partial\psi}{\partial y}+x\right)\left(\frac{dx}{dS}\right)=0$$

Thus each problem of torsion is reduced to the problem of finding a function y satisfying equation and the boundary condition given above.

Stress function method: (Prandtl approach)

As in the case of beams, the torsion problem formulated above is commonly solved by introducing a single stress function. This procedure has the advantage of leading to simpler boundary conditions as compared to Equation given above. The method is proposed by **Prandtl**. In this method, the principal unknowns are the stress components rather than the displacement components as in the previous approach.

Based on the result of the torsion of the circular shaft, let the non-vanishing components be τ_{zx} and τ_{yz}. The remaining stress components σ_x, σ_y and σ_z and τ_{xy} are assumed to be zero. In order to satisfy the equations of equilibrium, we should have

$$\frac{\partial\tau_{xz}}{\partial z}=0,\quad \frac{\partial\tau_{yz}}{\partial z}=0,\quad \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}=0$$

The first two are already satisfied since τ_{xz} and τ_{yz}, as given by Equations (d) and (e) are independent of z.

In order to satisfy the third condition, we assume a function $\phi(x, y)$ called Prandtl stress function such that

$$\tau_{xz}=\frac{\partial\phi}{\partial y},\quad \tau_{yz}=-\frac{\partial\phi}{\partial x}$$

With this stress function, (called Prandtl torsion stress function), the third condition is also satisfied. The assumed stress components, if they are to be proper elasticity solutions, have to satisfy the compatibility conditions. We can substitute these directly into the stress equations of compatibility. Alternately, we can determine the strains corresponding to the assumed stresses and then apply the strain compatibility conditions.

Therefore from above Equations , we have

$$\frac{\partial\phi}{\partial y}=G\theta\left(\frac{\partial\psi}{\partial x}-y\right) \qquad -\frac{\partial\phi}{\partial x}=G\theta\left(\frac{\partial\psi}{\partial y}+x\right)$$

Eliminating ψ by differentiating the first with respect to y, the second with respect to x, and subtracting from the first, we find that the stress function must satisfy the differential equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-2G\theta$$

or $$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=H$$

where $H = -2G\theta$

The boundary condition becomes, introducing above Equation.

$$\frac{\partial\phi}{\partial y}\frac{dy}{dS}+\frac{\partial\phi}{\partial x}\frac{dx}{dS}=\frac{d\phi}{dS}=0$$

This shows that the stress function f must be constant along the boundary of the cross section. In the case of singly connected sections, example, for solid bars, this constant can be arbitrarily chosen. Since the stress components depend only on the differentials of ϕ, for a simply connected region, no loss of generality is involved in assuming ϕ = 0 on S. However, for a multi-connected region, example shaft having holes, certain additional conditions of compatibility are imposed. Thus the determination of stress distribution over a cross-section of a twisted bar is used in finding the function f that satisfies above Equation and is zero at the boundary.

Conditions at the Ends of the Twisted bar

On the two end faces, the resultants in x and y directions should vanish, and the moment about A should be equal to the applied torque M_t. The resultant in the x-direction is

$$\iint \tau_{xz}\,dxdy=\iint\frac{\partial\phi}{\partial y}dxdy=\int dx\int\frac{\partial\phi}{\partial y}dy$$

Therefore, $$\iint \tau_{xz}\,dxdy=0$$

Since ϕ is constant around the boundary. Similarly, the resultant in the y-direction is

$$\iint \tau_{yz}\,dxdy=-\iint\frac{\partial\phi}{\partial x}dxdy$$

$$=-\int dy\int\frac{\partial\phi}{\partial x}dx$$

hence, $\iint \tau_{yz} dxdy = 0$

Thus the resultant of the forces distributed over the ends of the bar is zero, and these forces represent a couple the magnitude of which is

$$M_t = \iint (x\tau_{yz} - y\tau_{xz})dxdy$$

$$= -\iint (x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y})dxdy$$

Therefore,

$$M_t = -\iint x\frac{\partial\phi}{\partial x}dxdy - \iint y\frac{\partial\phi}{\partial y}dxdy$$

Integrating by parts, and observing that $\phi = 0$ at the boundary, we get

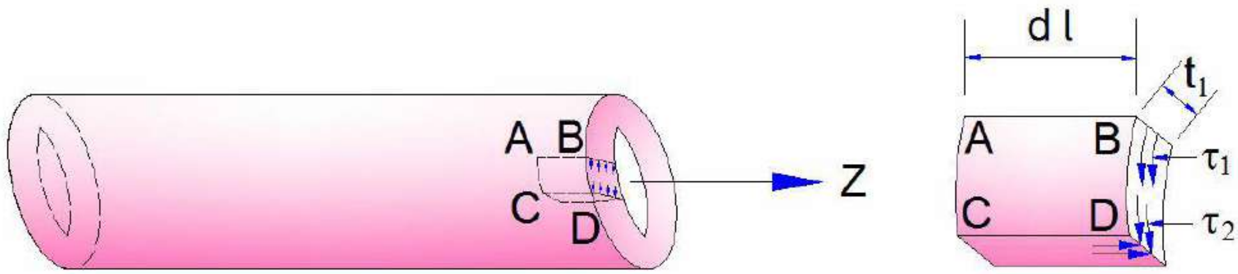
$$M_t = \iint \phi dxdy + \iint \phi dxdy$$

$$\therefore M_t = 2\iint \phi dxdy$$

Hence, we observe that each of the integrals in Equation contributing one half of the torque due to t_{xz} and the other half due to t_{yz}. Thus all the differential equations and boundary conditions are satisfied if the stress function ϕ obeys above Equations and the solution obtained in this manner is the exact solution of the torsion problem.

Torsion of thin walled open and closed sections

Consider a thin-walled tube subjected to torsion. The thickness of the tube may not be uniform as shown in the Figure below.



Torsion of thin walled sections

Since the thickness is small and the boundaries are free, the shear stresses will be essentially parallel to the boundary. Let τ be the magnitude of shear stress and t is the thickness.

Now, consider the equilibrium of an element of length D l as shown in Figure below. The areas of cut faces *AB* and *CD* are t_1 D l and t_2 D l respectively. The shear stresses (complementary shears) are t_1 and t_2.

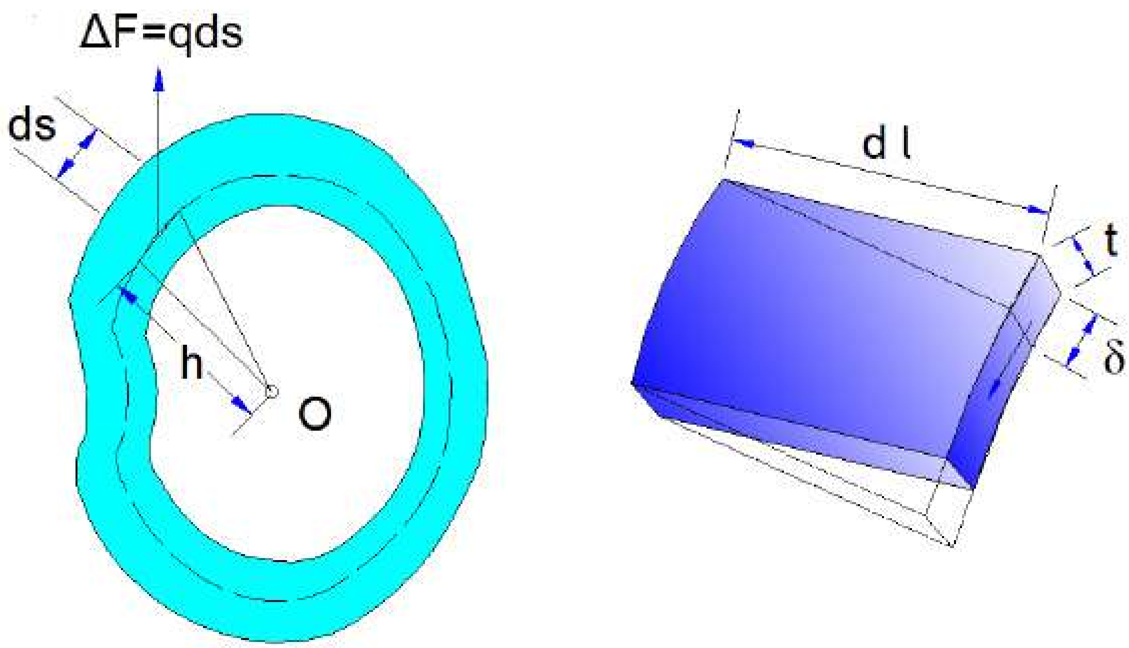
For equilibrium in z-direction, we have

$-\tau_1 t_1 \Delta l + \tau_2 t_2 \Delta l = 0$

Therefore, $\tau_1 t_1 = \tau_2 t_2 = q = \text{constant}$

Hence the quantity $\tau\, t$ is constant. This is called the shear flow q, since the equation is similar to the flow of an incompressible liquid in a tube of varying area.

Determination of Torque due to shear and Rotation:



Cross section of a thin-walled tube and torque due to shear

Consider the torque of the shear about point O

The force acting on the elementary length dS of the tube = $\Delta F = \tau\, t\, dS = q\, dS$

The moment arm about O is h and hence the torque = $\Delta M_t = (q dS)\, h$

Therefore, $\Delta M_t = 2qdA$

where dA is the area of the triangle enclosed at O by the base dS.

Hence the total torque is

$M_t = \Sigma\, 2qdA+$

Therefore, $M_t = 2qA$

Where A is the area enclosed by the centre line of the tube

To determine the twist of the tube

In order to determine the twist of the tube, Castigliano's theorem is used. Referring to Figure 7.7(b), the shear force on the element is $\tau\, t\, dS = q dS$. Due to shear strain γ, the force does work equal to ΔU

i.e., $$\Delta U = \frac{1}{2}(\tau\, t dS)\delta$$

$$= \frac{1}{2}(\tau\, t dS)\gamma.\Delta l$$

$$= \frac{1}{2}(\tau\, t dS).\Delta l.\frac{\tau}{G} \quad (\text{since } \tau = G\gamma)$$

$$= \frac{\tau^2 t^2 dS \Delta l}{2Gt}$$

$$= \frac{q^2 dS \Delta l}{2Gt}$$

$$= \frac{q^2 \Delta l}{2G}.\frac{dS}{t}$$

$$\Delta U = \frac{M_t^2 \Delta l}{8A^2 G}.\frac{dS}{t}$$

Therefore, the total elastic strain energy is

$$U = \frac{M_t^2 \Delta l}{8A^2 G}\oint \frac{dS}{t}$$

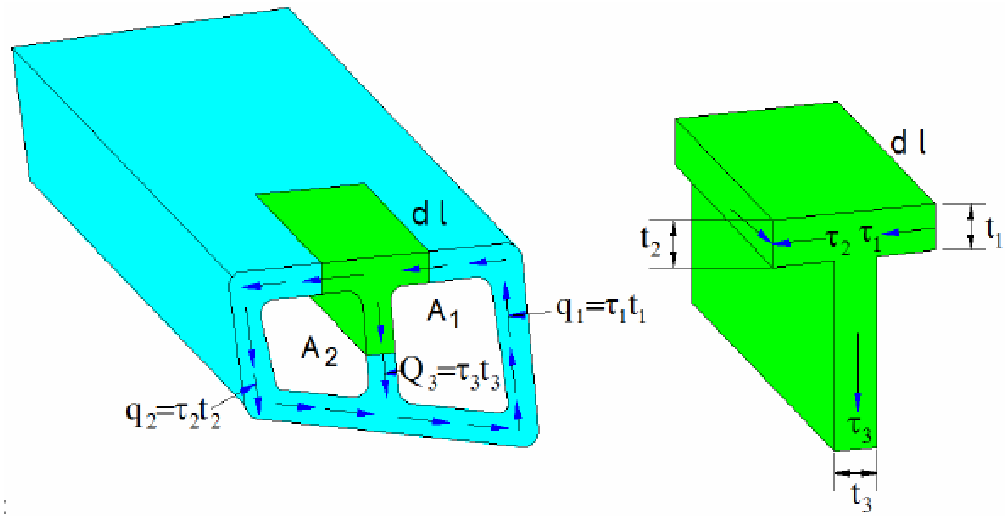
Hence, the twist or the rotation per unit length ($\Delta l = 1$) is

$$\theta = \frac{\partial U}{\partial M_t} = \frac{M_t}{4A^2 G}\oint \frac{dS}{t}$$

or $$\theta = \frac{2qA}{4A^2 G}\oint \frac{dS}{t}$$

or $$\theta = \frac{q}{2AG}\oint \frac{dS}{t}$$

Torsion of thin walled multiple cell closed section



Torsion of thin-walled multiple cell closed section

Consider the two-cell section shown in the Figure 7.8. Let A_1 and A_2 be the areas of the cells 1 and 2 respectively. Consider the equilibrium of an element at the junction as shown in the Figure 7.8(b). In the direction of the axis of the tube, we can write

$$-\tau_1 t_1 \ \Delta l + \tau_2 t_2 \ \Delta l + \tau_3 t_3 \ \Delta l \ = 0$$

or $\tau_1 t_1 = \ \tau_2 t_2 + \ \tau_3 t_3$

i.e., $q_1 = q_2 + q_3$

This is again equivalent to a fluid flow dividing itself into two streams. Now, choose moment axis, such as point O as shown in the Figure 7.9.

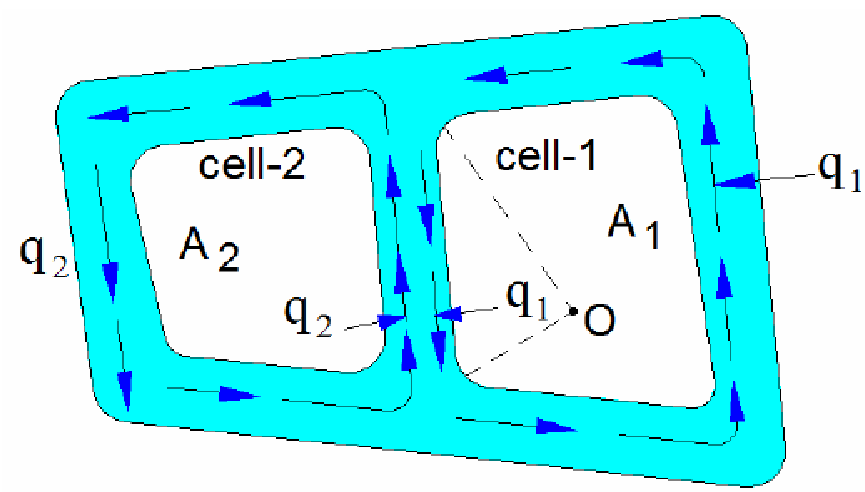


Figure. 7.9 Section of a thin walled multiple cell beam and moment axis

The shear flow in the web is considered to be made of q_1 and $-q_2$, since $q_3 = q_1 - q_2$.
Moment about O due to q_1 flowing in cell 1 (including web) is

$M_{t_1} = 2q_1A_1$

Similarly, the moment about O due to q_2 flowing in cell 2 (including web) is

$M_{t_2} = 2q_2 \ (A_2{+}A_1) - 2q_2A_1$

The second term with the negative sign on the right hand side is the moment due to shear flow q_2 in the middle web.

Therefore, The total torque is

$M_t = M_{t_1} + M_{t_2}$

$M_t = 2q_1A_1 + 2q_2A_2$ (a)

To Find the Twist (θ)

For continuity, the twist of each cell should be the same.

We have

$$\theta = \frac{q}{2AG}\oint\frac{dS}{t}$$

or $$2G\theta = \frac{1}{A}\int\frac{qdS}{t}$$

Let $a_1 = \oint\frac{dS}{t}$ for Cell 1 including the web

$a_2 = \oint\frac{dS}{t}$ for Cell 2 including the web

$a_{12} = \oint\frac{dS}{t}$ for the web only

Then for Cell 1

$$2G\theta = \frac{1}{A_1}(a_1 q_1 - a_{12} q_2) \qquad \text{(b)}$$

For Cell 2

$$2G\theta = \frac{1}{A_2}(a_2 q_2 - a_{12} q_1) \qquad \text{(c)}$$

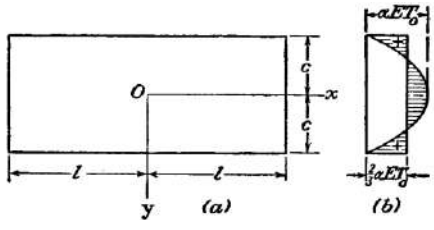
Equations (a), (b) and (c) are sufficient to solve for q_1, q_2 and θ.

Thermal stress.

The simplest case of Thermal stress distribution:

the causes of initial stresses in a body is nonuniform heating. With rising temperature the elements of a body expand. Such an expansion generally cannot proceed freely in a continuous body, and stresses due to the heating are set up. In many cases of machine design, such as in the design of steam turbines and Diesel engines, *thermal stresses* are of great practical importance and must be considered in more detail.

The simpler problems of thermal stress can easily be reduced to problems of boundary force of types already considered. As a first example



let us consider a thin rectangular plate of uniform thickness in which the temperature T is an even function of y (Fig. 222) and is independent of x and z. The longitudinal thermal expansion αT will be entirely suppressed by applying to each element of the plate the longitudinal compressive stress

$$\sigma_x' = -\alpha TE \tag{a}$$

Since the plate is free to expand laterally the application of the stresses (a) will not produce any stresses in the lateral directions and to maintain the stresses (a) throughout the plate it will be necessary to distribute compressive forces of the magnitude (a) at the ends of the plate only. These compressive forces will completely suppress any expansion of the plate in the direction of the x-axis due to the temperature rise T. To get the *thermal stresses* in the plate, which is free from external forces, we have to superpose on the stresses (a) the stresses produced in the plate by tensile forces of intensity αTE distributed at the ends. These forces have the resultant

$$\int_{-c}^{+c} \alpha TE\, dy$$

and at a sufficient distance from the ends they will produce approximately uniformly distributed tensile stress of the magnitude

$$\frac{1}{2c}\int_{-c}^{+c} \alpha TE\, dy$$

so that the thermal stresses in the plate with free ends at a considerable distance from the ends will be

$$\sigma_x = \frac{1}{2c}\int_{-c}^{+c} \alpha TE\, dy - \alpha TE \tag{b}$$

Assuming, for example, that the temperature is distributed parabolically and is given by the equation

$$T = T_0\left(1 - \frac{y^2}{c^2}\right)$$

we get, from Eq. (*b*),

$$\sigma_x = \frac{2}{3}\alpha T_0 E - \alpha T_0 E\left(1 - \frac{y^2}{c^2}\right) \tag{c}$$

This stress distribution is shown in Fig. 222*b*. Near the ends the stress distribution produced by the tensile forces is not uniform and can be calculated by the method explained on page 167. Superposing these stresses on the compressive stresses (*a*), the thermal stresses near the end of the plate will be obtained.

If the temperature T is not symmetrical with respect to the x-axis, we begin again with compressive stresses (*a*) suppressing the strain ϵ_x. In the nonsymmetrical cases these stresses give rise not only to a resultant force $-\int_{-c}^{+c} \alpha ET\,dy$ but also to a resultant couple $-\int_{-c}^{+c} \alpha ETy\,dy$, and in order to satisfy the conditions of equilibrium we must superpose on the compressive stresses (*a*) a uniform tension, determined as before, and bending stresses $\sigma''_x = \sigma y/c$ determined from the condition that the moment of the forces distributed over a cross section must be zero. Then

$$\int_{-c}^{+c} \frac{\sigma y^2\,dy}{c} - \int_{-c}^{+c} \alpha ETy\,dy = 0$$

from which

$$\frac{\sigma}{c} = \frac{3}{2c^3}\int_{-c}^{+c} \alpha ETy\,dy, \qquad \sigma_x'' = \frac{3y}{2c^3}\int_{-c}^{+c} \alpha ETy\,dy$$

Then the total stress is

$$\sigma_x = -\alpha ET + \frac{1}{2c}\int_{-c}^{+c} \alpha ET\,dy + \frac{3y}{2c^3}\int_{-c}^{+c} \alpha ETy\,dy \tag{d}$$

In this discussion it was assumed that the plate was thin in the z-direction. Suppose now that the dimension in the z-direction is large. We have then a plate with the xz-plane as its middle plane, and a thickness $2c$. Let the temperature T be, as before, independent of x and z, and so a function of y only.

The free thermal expansion of an element of the plate in the x- and z-directions will be completely suppressed by applying stresses σ_x, σ_z obtained from Eqs. (3), page 7, by putting $\epsilon_x = \epsilon_z = -\alpha T$, $\sigma_y = 0$. These equations then give

$$\sigma_x = \sigma_z = -\frac{\alpha E T}{1 - \nu} \tag{e}$$

The elements can be maintained in this condition by applying the distributions of compressive force given by (e) to the edges (x = constant, z = constant). The thermal stress in the plate free from external force is obtained by superposing on the stresses (e) the stresses due to application of equal and opposite distributions of force on the edges. If T is an even function of y such that the mean value over the thickness of the plate is zero, the resultant force per unit run of edge is zero, and by Saint-Venant's principle (Art. 18) it produces no stress except near the edge.

If the mean value of T is not zero, uniform tensions in the x- and z-directions corresponding to the resultant force on the edge must be superposed on the compressive stresses (e). If in addition to this the temperature is not symmetrical with respect to the xz-plane, we must add the bending stresses. In this manner we finally arrive at the equation

$$\sigma_x = \sigma_z = -\frac{\alpha T E}{1 - \nu} + \frac{1}{2c(1 - \nu)} \int_{-c}^{+c} \alpha T E \, dy + \frac{3y}{2c^3(1 - \nu)} \int_{-c}^{+c} \alpha T E y \, dy \tag{f}$$

which is analogous to the Eq. (d) obtained before. By using Eq. (f) we can easily calculate thermal stresses in a plate, if the distribution of temperature T over the thickness of the plate is known.

Consider, as an example, a plate which has initially a uniform temperature T_0 and which is being cooled down by maintaining the surfaces $y = \pm c$ at a constant temperature T_1.[1] By Fourier's theory the distribution of temperature at any instant t is

$$T = T_1 + \frac{4}{\pi} (T_0 - T_1) \left(e^{-p_1 t} \cos \frac{\pi y}{2c} - \frac{1}{3} e^{-p_3 t} \cos \frac{3\pi y}{2c} + \cdots \right) \tag{g}$$

in which $p_1, p_3 = 3^2p_1, \ldots, p_n = n^2p_1, \ldots$, are certain constants. Substituting in Eq. (f), we find

$$\sigma_x = \sigma_z = \frac{4\alpha E(T_0 - T_1)}{\pi(1-\nu)} \left[e^{-p_1t}\left(\frac{2}{\pi} - \cos\frac{\pi y}{2c}\right) + \frac{1}{3}e^{-p_3t}\left(\frac{2}{3\pi} + \cos\frac{3\pi y}{2c}\right) + \frac{1}{5}e^{-p_5t}\left(\frac{2}{5\pi} - \cos\frac{5\pi y}{2c}\right) + \cdots \right] \quad (h)$$

After a moderate time the first term acquires dominant importance, and we can assume

$$\sigma_x = \sigma_z = \frac{4\alpha E(T_0 - T_1)}{\pi(1-\nu)} \cdot e^{-p_1t}\left(\frac{2}{\pi} - \cos\frac{\pi y}{2c}\right)$$

For $y = \pm c$ we have tensile stresses

$$\sigma_x = \sigma_z = \frac{4\alpha E(T_0 - T_1)}{\pi(1-\nu)} e^{-p_1t}\frac{2}{\pi}$$

At the middle plane $y = 0$ we obtain compressive stresses

$$\sigma_x = \sigma_z = -\frac{4\alpha E(T_0 - T_1)}{\pi(1-\nu)} e^{-p_1t}\left(1 - \frac{2}{\pi}\right)$$

The points with zero stresses are obtained from the equation

$$\frac{2}{\pi} - \cos\frac{\pi y}{2c} = 0$$

from which

$$y = \pm 0.560c$$

If the surfaces $y = \pm c$ of a plate are maintained at two different temperatures T_1, T_2, a steady state of heat flow is established after a certain time and the temperature is then given by the linear function

$$T = \frac{1}{2}(T_1 + T_2) + \frac{1}{2}(T_1 + T_2)\frac{y}{c} \quad (i)$$

Substitution in Eq. (f) shows that the thermal stresses are zero,[1] provided, of course, that the plate is not restrained. If the edges are perfectly restrained against expansion and rotation, the stress induced by the heating is given by Eqs. (e). For instance if $T_2 = -T_1$ we have from (i)

$$T = T_1\frac{y}{c} \quad (j)$$

and Eqs. (e) give

$$\sigma_x = \sigma_z = -\frac{\alpha T}{1 - \nu} T_1 \frac{y}{c} \tag{k}$$

The maximum stress is

$$(\sigma_x)_{\text{max.}} = (\sigma_z)_{\text{max.}} = \frac{\alpha E T_1}{1 - \nu} \tag{l}$$

The thickness of the plate does not enter in this formula, but in the case of a thicker plate a greater difference of temperature between the two surfaces usually exists. Thus a thick plate of a brittle material is more liable to break due to thermal stresses than a thin one.

As a last example let us consider a sphere of large radius and assume that there occurs a temperature rise T in a small spherical element of radius a at the center of the large sphere. Since the element is not free to expand a pressure p will be produced at the surface of the element. The radial and the tangential stresses due to this pressure at any point of the sphere at a radius $r > a$ can be calculated from formulas (197) and (198) (see page 359). Assuming the outer radius of the sphere as very large in comparison with a we obtain from these formulas

$$\sigma_r = -\frac{pa^3}{r^3}, \qquad \sigma_t = \frac{pa^3}{2r^3} \tag{m}$$

At the radius $r = a$ we obtain

$$\sigma_r = -p, \qquad \sigma_t = \tfrac{1}{2}p$$

and the increase of this radius, due to pressure p, is

$$\Delta r = (a\epsilon_t)_{r=a} = \frac{a}{E}[\sigma_t - \nu(\sigma_r + \sigma_t)]_{r=a} = \frac{pa}{2E}(1 + \nu)$$

This increase must be equal to the increase of the radius of the heated spherical element produced by temperature rise and pressure p. Thus we obtain the equation

$$\alpha Ta - \frac{pa}{E}(1 - 2\nu) = \frac{pa}{2E}(1 + \nu)$$

from which

$$p = \frac{2}{3}\frac{\alpha TE}{1 - \nu} \tag{n}$$

Substituting in equations (m) we obtain the formulas for the stresses outside the heated element

$$\sigma_r = -\frac{2}{3}\frac{\alpha TEa^3}{(1 - \nu)r^3}, \qquad \sigma_t = \frac{1}{3}\frac{\alpha TEa^3}{(1 - \nu)r^3} \tag{o}$$

Module-IV

Theoretical concepts of plasticity

The classical **theory of plasticity** grew out of the study of metals in the late nineteenth century. It is concerned with materials which initially deform elastically, but which deform plastically upon reaching a yield stress. In metals and other crystalline materials the occurrence of plastic deformations at the micro-scale level is due to the motion of dislocations and the migration of grain boundaries on the micro-level. In sands and other granular materials plastic flow is due both to the irreversible rearrangement of individual particles and to the irreversible crushing of individual particles. Similarly, compression of bone to high stress levels will lead to particle crushing. The deformation of microvoids and the development of micro-cracks is also an important cause of plastic deformations in materials such as rocks.

A good part of the discussion in what follows is concerned with the plasticity of metals; this is the 'simplest' type of plasticity and it serves as a good background and introduction to the modelling of plasticity in other material-types. There are two broad groups of metal plasticity problem which are of interest to the engineer and analyst. The first involves relatively small plastic strains, often of the same order as the elastic strains which occur. Analysis of problems involving small plastic strains allows one to design structures optimally, so that they will not fail when in service, but at the same time are not stronger than they really need to be. In this sense, plasticity is seen as a material failure1.

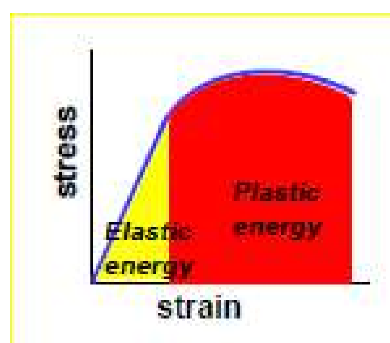
The second type of problem involves very large strains and deformations, so large that the elastic strains can be disregarded. These problems occur in the analysis of metals manufacturing and forming processes, which can involve extrusion, drawing, forging, rolling and so on. In these latter-type problems, a simplified model known as perfect plasticity is usually employed (see below), and use is made of special limit theorems which hold for such models.

Plastic deformations are normally rate independent, that is, the stresses induced are independent of the rate of deformation (or rate of loading). This is in marked contrast to classical Newtonian fluids for example, where the stress levels are governed by the rate of deformation through the viscosity of the fluid.

Materials commonly known as “plastics” are not plastic in the sense described here. They, like other polymeric materials, exhibit viscoelastic behaviour where, as the name suggests, the material response has both elastic and viscous components. Due to their viscosity, their response is, unlike the plastic materials, rate-dependent. Further, although the viscoelastic materials can suffer irrecoverable deformation, they do not have any critical yield or threshold stress, which is the characteristic property of plastic behaviour. When a material undergoes plastic deformations, i.e. irrecoverable and at a critical yield stress, and these effects are rate dependent, the material is referred to as being viscoplastic.

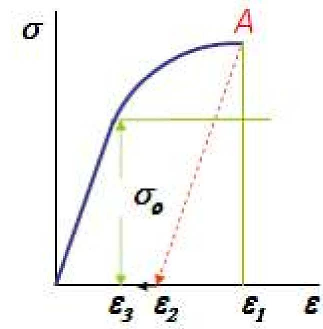
Plasticity theory began with Tresca in 1864, when he undertook an experimental program into the extrusion of metals and published his famous yield criterion discussed later on. Further advances with yield criteria and plastic flow rules were made in the years which followed by Saint-Venant, Levy, Von Mises, Hencky and Prandtl. The 1940s saw the advent of the classical theory; Prager, Hill, Drucker and Koiter amongst others brought together many fundamental aspects of the theory into a single framework. The arrival of powerful computers in the 1980s and 1990s provided the impetus to develop the theory further, giving it a more rigorous foundation based on thermodynamics principles, and brought with it the need to consider many numerical and computational aspects to the plasticity problem.

- Plastic deformation is a non reversible process where Hooke’s law is no longer valid.
- One aspect of plasticity in the viewpoint of structural design is that it is concerned with predicting the maximum load, which can be applied to a body without causing excessive yielding.
- Another aspect of plasticity is about the plastic forming of metals where large plastic required to



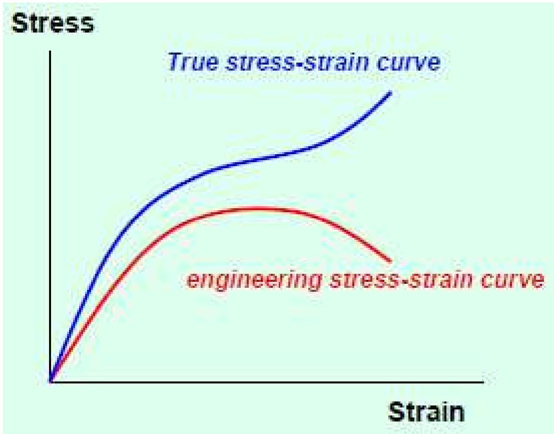
Plastic and elastic deformation in uniaxial tension

- True stress-strain curve for typical ductile materials, i.e., aluminium, show that the stress - strain relationship follows up the Hooke's law up to the yield point, σ_o
- Beyond σ_o, the metal deforms plastically with strain-hardening. This cannot be related by any simple constant of proportionality.
- If the load is released from straining up to point A, the total strain will immediately decrease from ε_1 to ε_2 by an amount of σ/E.
- The strain ε_1-ε_2 is the recoverable elastic strain. Also there will be a small amount of the plastic strain ε_2-ε_3 known as anelastic behaviour which will disappear by time.(neglected in plasticity theories.)



Typical true stress-strain curves for a ductile metal.

- The engineering stress – strain curve is based entirely on the original dimensions of the specimen (this cannot represent true deformation characteristic of the material).
- The true stress – strain curve is based on the instantaneous specimen dim ensions



Engineering stress-strain and true stress-strain curves.

Yield criteria

Commencement of plastic deformation in materials is predicted by yield criteria. Yield criteria are also called theories of yielding. A number of yield criteria have been developed for ductile and brittle materials.

Tresca yield criterion:

It states that when the maximum shear stress within an element is equal to or greater than a critical value, yielding will begin.

$\tau_{max} \geq k$

Where k is shear yield strength.

Or $\tau_{max} = (\sigma_1 - \sigma_3)/2 = k$ where σ_1 and σ_3 are principal stresses

Or $\sigma_1 - \sigma_3 = Y$

For uniaxial tension, we have k = Y/2

Here Y or k are material properties. The intermediate stress σ_2 has no effect on yielding.

Von Mises criterion

According to this criterion yielding occurs when

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$$

For plane strain condition, we have: $\sigma_2 = (\sigma_1 + \sigma_3)/2$

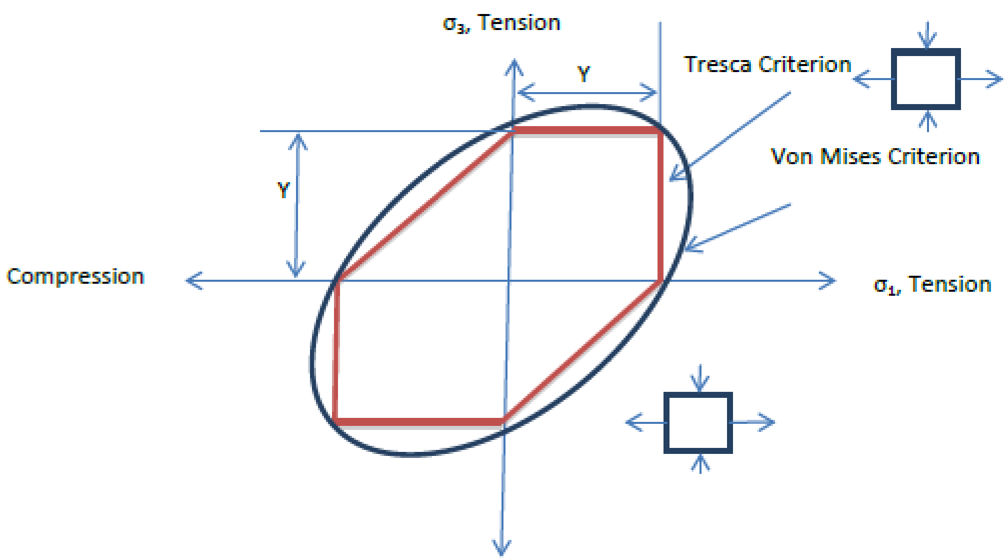
Hence, from the distortion energy criterion, we have $\sigma_1 - \sigma_3 = \frac{2}{\sqrt{3}}Y$ Here, $\frac{2}{\sqrt{3}}Y$ is called plane strain yield strength. Von Mises criterion can also be interpreted as the yield criterion which states that when octahedral shear stress reaches critical value, yielding commences.

The octahedral shear stress is the shear stresses acting on the faces of an octahedron, given by:

$$\tau_{oct} = \frac{1}{3}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2}$$

According to Tresca criteria we know, $(\sigma_1 - \sigma_3)/2 = k$. Therefore,

$k = \frac{Y}{\sqrt{3}}$



Yield loci for two yield criteria in plane stress

Von Mises yield criterion is found to be suitable for most of the ductile materials used in forming operations. More often in metal forming, this criterion is used for the analysis. The suitability of the yield criteria has been experimentally verified by conducting torsion test on thin walled tube, as the thin walled tube ensures plane stress. However, the use of Tresca criterion is found to result in negligible difference between the two criteria. We observe that the von Mises criterion is able to predict the yielding independent of the sign of the stresses because this criterion has square terms of the shear stresses.

Effective stress and effective strain:

Effective stress is defined as that stress which when reaches critical value, yielding can commence.

For Tresca criterion, effective stress is $\sigma_{eff} = \sigma_1 - \sigma_3$

For von Mises criterion, the effective stress is

$$\frac{1}{\sqrt{2}}\{ [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]\}^{1/2}$$

The factor $1/\sqrt{2}$ is chosen such that the effective stress for uniaxial tensile loading is equal to uniaxial yield strength Y.

The corresponding effective strain is defined as:

$$\varepsilon_{eff} = \frac{2}{3}\ (\varepsilon_1 - \varepsilon_3)$$

From von Mises criterion:

$$\text{Effective strain} = (\sqrt{2}/3)\{ [(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2]\}^{1/2}$$

For Tresca:

Effective strain = $(2/3)(\varepsilon_1 - \varepsilon_3)$

For uniaxial loading, the effective strain is equal to uniaxial tensile strain.

Note: The constants in effective strain expressions, given above are chosen so that for uniaxial loading, the effective strain reduces to uniaxial strain.

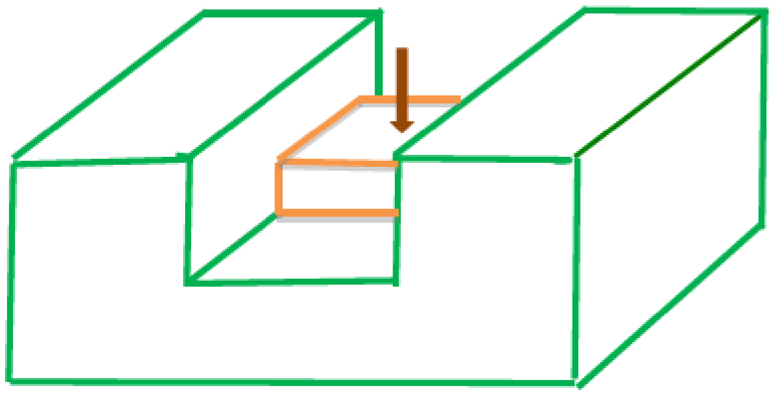
Normal strain versus shear strain:

We know for pure shear: $\sigma_1 = -\sigma_3$ and $\sigma_1 = \tau$

Therefore from the effective stress equation of Tresca we get: Effective stress = $2\sigma_1 = 2\tau_1$

Similarly using von Mises effective stress, we have

Effective stress = $\sqrt{3}\sigma_1 = \sqrt{3}\tau_1$



A plane strain compression forging process

Plastic stress strain relationship,

Elastic plastic problems in bending and torsion.